Additive Number Theory and Automata

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Additive number theory

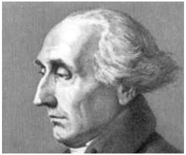
Let S, T be subsets of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The **principal problem** of additive number theory is to determine whether every element of T (or every sufficiently large element of T) can be written as the sum of some **constant** number of elements of S, not necessarily distinct.

Often (but not always) $T = \mathbb{N}$ and S is a relatively sparse subset of $\mathbb{N}$.

Lagrange's theorem

Probably the most famous example is **Lagrange's theorem** (1770):



(a) every natural number is the sum of four squares; and

(b) three squares do not suffice for numbers of the form $4^a(8k+7)$.

(Conjectured by Bachet in 1621.)

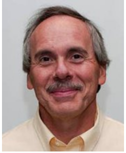
Goldbach's conjecture

Let $\mathbb{P} = \{2, 3, 5, \ldots, \}$ be the prime numbers.

Goldbach's conjecture (1742): every even number ≥ 4 is the sum of two primes.

So here $T = 2\mathbb{N}$.

Zwillinger's conjecture (1979): every even number > 4208 is the sum of 2 numbers, each of which is part of a twin-prime pair.



Additive bases

Let $S \subseteq \mathbb{N}$.

We say that a subset S is an **basis of order** h if every natural number can be written as the sum of h elements of S, not necessarily distinct.

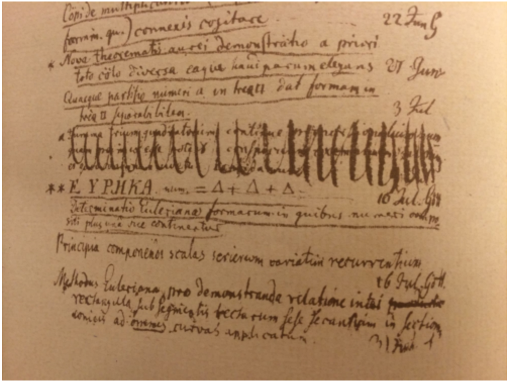
We say that a subset S is an **asymptotic basis of order** h if every *sufficiently large* natural number can be written as the sum of h elements of S, not necessarily distinct.

Usual convention: $0 \in S$.

Gauss's theorem for triangular numbers

A *triangular number* is a number of the form $n(n+1)/2$.

Gauss wrote the following in his diary on July 10 1796:



Coni de multiplic[illegible]
formarum qu.) connexis cogitare 22 Jun.

* Nova theorematis aurei demonstratio a priori
toto coelo diversa eaque haud parum elegans 27 Jun.

Quaecunque partitio numeri a in [illegible] dat formam in
tria □ separabilem. 3 Jul.

* ~~Forma trium quadratorum [illegible]~~
[illegible]

** ΕΥΡΗΚΑ num. $= \Delta + \Delta + \Delta$ 10 Jul. Gött.

Determinatio Euleriana formarum in quibus numeri compositi plus una vice continentur

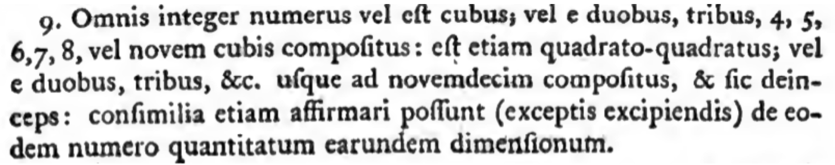
Principia componendi scalas serierum variatim recurrentium 16 Jul. Gött.

Methodus Euleriana pro demonstranda relatione inter [illegible] rectangula sub segmentis rectarum sese secantium in sectionibus conicis ad omnes curvas applicatur 31 Jul.

i.e., **The triangular numbers form an additive basis of order** 3

Waring's problem for powers

Edward Waring (1770) asserted,
without proof, that
every natural number is
– the sum of 4 squares
– the sum of 9 cubes
– the sum of 19 fourth powers
– "and so forth".



9. Omnis integer numerus vel eſt cubus; vel e duobus, tribus, 4, 5, 6,7, 8, vel novem cubis compoſitus: eſt etiam quadrato-quadratus; vel e duobus, tribus, &c. uſque ad novemdecim compoſitus, & ſic deinceps: conſimilia etiam affirmari poſſunt (exceptis excipiendis) de eodem numero quantitatum earundem dimenſionum.

Waring's problem

Let $g(k)$ be the least natural number m such that every natural number is the sum of m k'th powers.

Let $G(k)$ be the least natural number m such that every sufficiently large natural number is the sum of m k'th powers.

Proving that $g(k)$ and $G(k)$ exist, and determining their values, is **Waring's problem**.

By Lagrange we know $g(2) = G(2) = 4$.

Hilbert proved in 1909 that $g(k)$ and $G(k)$ exist for all k. (Later, proof simplified by Vinogradov.)

By Wieferich and Kempner we know $g(3) = 9$.

We know that $4 \leq G(3) \leq 7$, but the true value is still unknown.

Other additive bases?

What other sets can be additive bases?

Not the powers of 2 – too sparse.

Need a set whose natural density is at least $N^{1/k}$ for some k.

But this is not enough: consider the set

$$S = \{2^{2^n} + i \;:\; n \geq 1 \text{ and } 0 \leq i < 2^n\}.$$

Its density is $\Omega(N^{1/2})$.

But S does not form an additive basis of any finite order, because adding k elements of S in decreasing order can only result in at most $2k+1$ "one" bits in the highest-order positions.

Palindromes

- How about numbers with palindromic base-b expansions?
- A *palindrome* is any string that is equal to its reversal
- Examples are `radar` (English), `ressasser` (French), and `10001`.
- We call a natural number a *base-b palindrome* if its base-b representation (without leading zeroes) is a palindrome
- Examples are $16 = [121]_3$ and $297 = [100101001]_2$.
- Binary palindromes ($b = 2$) form sequence A006995 in the *On-Line Encyclopedia of Integer Sequences* (OEIS):
  $$0, 1, 3, 5, 7, 9, 15, 17, 21, 27, 31, 33, 45, 51, 63, \ldots$$
- They have density $\Theta(N^{1/2})$.

The problem

Do the base-b palindromes form an additive basis, and if so, of what order?

William Banks (2015) showed that every natural number is the sum of at most 49 base-10 palindromes.
(*INTEGERS* **16** (2016), #A3)

Javier Cilleruelo, Florian Luca, and Lewis Baxter (2017) showed that for all bases $b \geq 5$, every natural number is the sum of three base-b palindromes.
(*Math. Comp.* (2017), to appear)

What we proved

However, the case of bases $b = 2, 3, 4$ was left unsolved. We proved

Theorem (Rajasekaran, JOS, Smith)

Every natural number N is the sum of 4 binary palindromes. The number 4 is optimal.

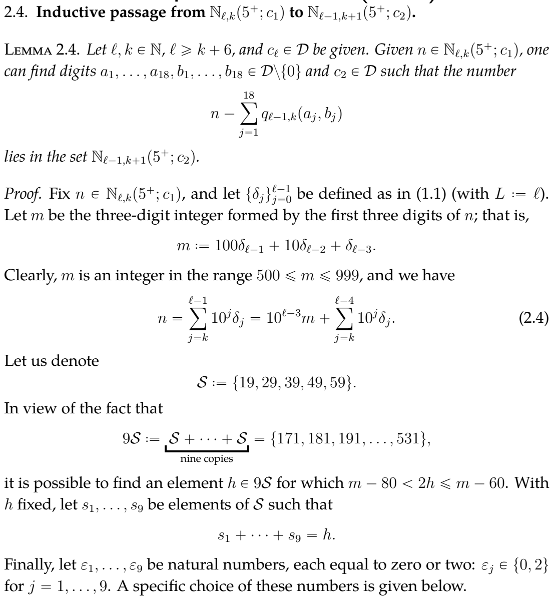
For example,

$$\begin{aligned} 10011938 &= 5127737 + 4851753 + 32447 + 1 \\ &= [10011100011111000111001]_2 \\ &+ [10010100000100000101001]_2 \\ &+ [111111010111111]_2 \\ &+ [1]_2. \end{aligned}$$

4 is optimal: 10011938 is not the sum of 2 binary palindromes.

Previous proofs were complicated (1)

Excerpt from Banks (2015):



2.4. **Inductive passage from** $\mathbb{N}_{\ell,k}(5^+;c_1)$ **to** $\mathbb{N}_{\ell-1,k+1}(5^+;c_2)$**.**

LEMMA 2.4. *Let $\ell, k \in \mathbb{N}$, $\ell \geqslant k+6$, and $c_\ell \in \mathcal{D}$ be given. Given $n \in \mathbb{N}_{\ell,k}(5^+;c_1)$, one can find digits $a_1, \dots, a_{18}, b_1, \dots, b_{18} \in \mathcal{D}\backslash\{0\}$ and $c_2 \in \mathcal{D}$ such that the number*

$$n - \sum_{j=1}^{18} q_{\ell-1,k}(a_j, b_j)$$

lies in the set $\mathbb{N}_{\ell-1,k+1}(5^+;c_2)$.

Proof. Fix $n \in \mathbb{N}_{\ell,k}(5^+;c_1)$, and let $\{\delta_j\}_{j=0}^{\ell-1}$ be defined as in (1.1) (with $L := \ell$). Let m be the three-digit integer formed by the first three digits of n; that is,

$$m := 100\delta_{\ell-1} + 10\delta_{\ell-2} + \delta_{\ell-3}.$$

Clearly, m is an integer in the range $500 \leqslant m \leqslant 999$, and we have

$$n = \sum_{j=k}^{\ell-1} 10^j \delta_j = 10^{\ell-3} m + \sum_{j=k}^{\ell-4} 10^j \delta_j. \tag{2.4}$$

Let us denote

$$\mathcal{S} := \{19, 29, 39, 49, 59\}.$$

In view of the fact that

$$9\mathcal{S} := \underbrace{\mathcal{S} + \cdots + \mathcal{S}}_{\text{nine copies}} = \{171, 181, 191, \dots, 531\},$$

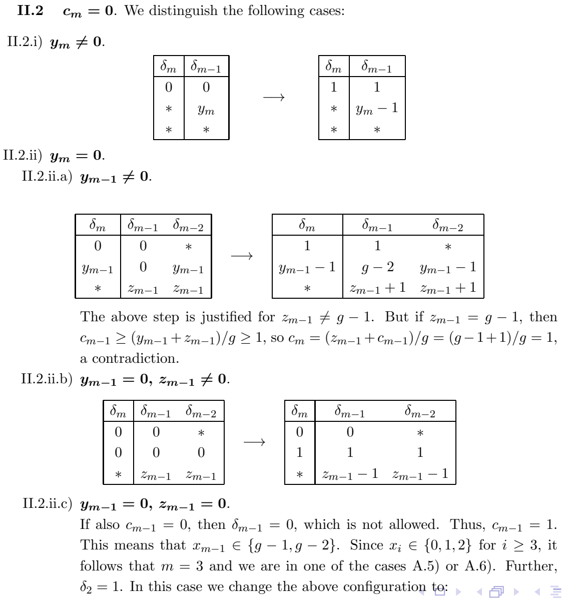
it is possible to find an element $h \in 9\mathcal{S}$ for which $m - 80 < 2h \leqslant m - 60$. With h fixed, let $s_1, \dots, s_9$ be elements of $\mathcal{S}$ such that

$$s_1 + \cdots + s_9 = h.$$

Finally, let $\varepsilon_1, \dots, \varepsilon_9$ be natural numbers, each equal to zero or two: $\varepsilon_j \in \{0, 2\}$ for $j = 1, \dots, 9$. A specific choice of these numbers is given below.

Previous proofs were complicated (2)

Excerpt from Cilleruelo et al. (2017)



II.2 $\boldsymbol{c_m = 0}$. We distinguish the following cases:

II.2.i) $\boldsymbol{y_m \neq 0}$.

δ_m	δ_{m-1}
0	0
$*$	y_m
$*$	$*$

$\longrightarrow$

δ_m	δ_{m-1}
1	1
$*$	$y_m - 1$
$*$	$*$

II.2.ii) $\boldsymbol{y_m = 0}$.

II.2.ii.a) $\boldsymbol{y_{m-1} \neq 0}$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
y_{m-1}	0	y_{m-1}
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
1	1	$*$
$y_{m-1} - 1$	$g - 2$	$y_{m-1} - 1$
$*$	$z_{m-1} + 1$	$z_{m-1} + 1$

The above step is justified for $z_{m-1} \neq g - 1$. But if $z_{m-1} = g - 1$, then $c_{m-1} \geq (y_{m-1} + z_{m-1})/g \geq 1$, so $c_m = (z_{m-1} + c_{m-1})/g = (g - 1 + 1)/g = 1$, a contradiction.

II.2.ii.b) $\boldsymbol{y_{m-1} = 0,\ z_{m-1} \neq 0}$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
0	0	0
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
1	1	1
$*$	$z_{m-1} - 1$	$z_{m-1} - 1$

II.2.ii.c) $\boldsymbol{y_{m-1} = 0,\ z_{m-1} = 0}$.

If also $c_{m-1} = 0$, then $\delta_{m-1} = 0$, which is not allowed. Thus, $c_{m-1} = 1$. This means that $x_{m-1} \in \{g - 1, g - 2\}$. Since $x_i \in \{0, 1, 2\}$ for $i \geq 3$, it follows that $m = 3$ and we are in one of the cases A.5) or A.6). Further, $\delta_2 = 1$. In this case we change the above configuration to:

Previous proofs were complicated (3)

- Proofs of Banks and Cilleruelo et al. were long and case-based
- Difficult to establish
- Difficult to understand
- Difficult to check, too: the original Cilleruelo et al. proof had some minor flaws that were only noticed when the proof was implemented as a `Python` program
- Idea: could we automate such proofs?

The main idea of our proof

- Construct a finite-state machine that takes natural numbers as input, expressed in the desired base
- Allow the machine to nondeterministically "guess" a representation of the input as a sum of palindromes
- The machine accepts an input if it verifies its guess
- Then use a decision procedure to establish properties about the language of representations accepted by this machine (e.g., universality)

Picking a machine model for palindromes

What machine model?

- it should be possible to check if the guessed summands are palindromes
  - can be done with a pushdown automaton (PDA)
- it should be possible to add the summands and compare to the input
  - can be done with a finite automaton (DFA or NFA)

However

- Can't add summands with these machine models unless they are guessed in parallel
- Can't check if summands are palindromes if they are wildly different in length & presented in parallel
- Universality is not decidable for PDA's

What to do?

Visibly pushdown automata (VPA)

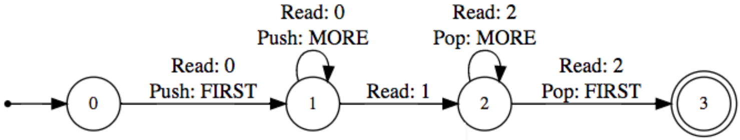
- Use *visibly-pushdown automata*!
- Popularized by Alur and Madhusudan in 2004, though similar ideas have been around for longer
- VPA's receive an input string, and read the string one letter at a time
- They have a (finite) set of states and a stack
- Upon reading a letter of the input string, the VPA can transition to a new state, and might modify the stack
- The states of the VPA are either *accepting* or *non-accepting*
- If the VPA can end up in an accepting state after it is done reading the input, then the VPA "accepts" the input, else it "rejects" it

Using the VPA's stack

- The VPA can only take very specific stack actions
- The input alphabet, Σ, is partitioned into three disjoint sets
  - Σ_c, the push alphabet
  - Σ_l, the local alphabet
  - Σ_r, the pop alphabet
- If the letter of the input string we read is from the push alphabet, the VPA pushes *exactly* one symbol onto its stack
- If the letter of the input string we read is from the pop alphabet, the VPA pops *exactly* one symbol off its stack
- If the letter of the input string we read is from the local alphabet, the VPA does not consult its stack at all

Example VPA

A VPA for the language $\{0^n 1 2^n \ : \ n \geq 1\}$:



The push alphabet is $\{0\}$, the local alphabet is $\{1\}$, and the pop alphabet is $\{2\}$.

Determinisation and Decidability

- A nondeterministic VPA can have several matching transition rules for a single input letter
- Nondeterministic VPA's are as powerful as deterministic VPA's
- VPL's are closed under union, intersection and complement. There are algorithms for all these operations.
- Testing emptiness, universality and language inclusion are decidable problems for VPA's
- But a nondeterministic VPA with n states can have as many as $2^{\Theta(n^2)}$ states when determinized!

Proof strategy

- We build a VPA that “guesses” inputs of roughly the same size, in parallel
- It checks to see that they are palindromes
- And it adds them together and verifies that the sum equals the input.
- There are some complications due to the VPA restrictions.

More details of the proof strategy

- To prove our result, we built 2 VPA's A and B:
  - A accepts all n-bit odd integers, $n \geq 8$, that are the sum of three binary palindromes of length either
    - n, $n-2$, $n-3$, or
    - $n-1$, $n-2$, $n-3$.
  - B accepts all valid representations of odd integers of length $n \geq 8$
- We then prove that all inputs accepted by B are accepted by A
- We used the ULTIMATE Automata Library
- Once A and B are built, we simply have to issue the command

```
assert(IsIncluded(B, A))
```

in ULTIMATE.

Bases 3 and 4

- Unfortunately, the VPA's for bases 3 and 4 are too large to handle in this way.
- So we need a different approach.
- Instead, we use ordinary nondeterministic finite automata (NFA).
- But they cannot recognize palindromes...
- Instead, we change the input representation so that numbers are represented in a "folded" way, where each digit at the beginning of its representation is paired with its corresponding digit at the end.
- With this we can prove...

Other results

Theorem

Every natural number $N > 256$ is the sum of at most three base-3 palindromes.

Theorem

Every natural number $N > 64$ is the sum of at most three base-4 palindromes.

This completes the classification for base-b palindromes for all $b \geq 2$.

An analogue of Lagrange's theorem

Using NFA's we can establish an analogue of Lagrange's four-square theorem.

- A *square* is any string that is some shorter string repeated twice
- Examples are `hotshots` (English), `couscous` (French), and 100100.
- We call an integer a *base-b square* if its base-b representation is a square
- Examples are $36 = [100100]_2$ and $3 = [11]_2$.
- The binary squares form sequence A020330 in the OEIS

$$3, 10, 15, 36, 45, 54, 63, 136, 153, 170, 187, 204, 221, \ldots$$

Lagrange's theorem strategy

Lemma.

(a) Every length-n integer, n odd, $n \geq 13$, is the sum of binary squares as follows: either

- one of length $n-1$ and one of length $n-3$, or
- two of length $n-1$ and one of length $n-3$, or
- one of length $n-1$ and two of length $n-3$, or
- one each of lengths $n-1$, $n-3$, and $n-5$, or
- two of length $n-1$ and two of length $n-3$, or
- two of length $n-1$, one of length $n-3$, and one of length $n-5$.

(b) Every length-n integer, n even, $n \geq 18$, is the sum of binary squares as follows: either

- two of length $n-2$ and two of length $n-4$, or
- three of length $n-2$ and one of length $n-4$, or
- one each of lengths n, $n-4$, and $n-6$, or
- two of length $n-2$, one of length $n-4$, and one of length $n-6$.

Lagrange's theorem

Note that

- Using automata we cannot state the theorem we *want* to prove
- This is due to the fact that we can't add squares of wildly differing lengths using the representation we chose
- But we *can* state the *stronger* result of the lemma on the previous slide
- So we are combining a decision procedure together with a heuristic search for an appropriate lemma to prove.

Results

Theorem. Every natural number $N > 686$ is the sum of at most 4 binary squares.

For example:

$$\begin{aligned} 10011938 &= 9291996 + 673425 + 46517 \\ &= [100011011100100011011100]_2 + [10100100011010010001]_2 \\ &+ [1011010110110101]_2 \end{aligned}$$

Here the 686 is optimal.

The list of all exceptions is

$$\begin{aligned} &1, 2, 4, 5, 7, 8, 11, 14, 17, 22, 27, 29, 32, 34, 37, 41, 44, 47, \\ &53, 62, 95, 104, 107, 113, 116, 122, 125, 131, 134, 140, 143, \\ &148, 155, 158, 160, 167, 407, 424, 441, 458, 475, 492, 509, \\ &526, 552, 560, 569, 587, 599, 608, 613, 620, 638, 653, 671, 686. \end{aligned}$$

Another result

Theorem

Every natural number is the sum of at most two binary squares and at most two powers of 2.

Generalizing: Waring's theorem for binary k'th powers

Recall Waring's theorem: *for every $k \geq 1$ there exists a constant $g(k)$ such that every natural number is the sum of $g(k)$ k'th powers of natural numbers.*

Could the same result hold for the binary k'th powers?

Two issues:

- 1 is not a binary k'th power, so it has to be "every sufficiently large natural number" and not "every natural number".
- The gcd g of the binary k'th powers need not be 1, so it actually has to be "every sufficiently large multiple of g".

gcd of the binary k'th powers

Theorem

The gcd *of the binary k'th powers is* $\gcd(k, 2^k - 1)$.

Example:

The binary 6'th powers are

$63, 2730, 4095, 149796, 187245, 224694, 262143, 8947848, 10066329, \ldots$

with gcd equal to $\gcd(6, 63) = 3$.

Waring's theorem for binary k'th powers

Theorem

Every sufficiently large multiple of $\gcd(k, 2^k - 1)$ *is the sum of a constant number (depending on* k*) of binary* k*'th powers.*

Obtained with Daniel Kane and Carlo Sanna.

Outline of the proof

Given a number N we wish to represent as a sum of binary k'th powers:

- choose a suitable power of 2, say 2^n, and express N as a polynomial in $x = 2^n$. We want $x^k \approx N$.
- use linear algebra to change the basis and instead express N as a linear combination of $c_k(n), c_k(n+1), \ldots, c_k(n+k-1)$ where

$$c_k(n) = \frac{2^{kn}-1}{2^n-1} = 1 + 2^n + 2^{2n} + \cdots + 2^{(k-1)n}.$$

- Such a linear combination would seem to provide an expression for N in terms of binary k'th powers, but there are three problems to overcome:
  (a) the coefficients of $c_k(i)$, $n \le i < n+k$, could be much too large;
  (b) the coefficients could be too small or negative;
  (c) the coefficients might not be integers.

All of these problems can be handled with some work...

The coefficients are too large

The linear combination is governed by the size of entries in the inverse of the *Vandermonde matrix* $M_k := V(1, 2, 4, \ldots, 2^{k-1})$, where

$$V(x_0, x_1, \ldots, x_{k-1}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{k-1} \\ x_0^2 & x_1^2 & \cdots & x_{k-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{k-1} & x_1^{k-1} & \cdots & x_{k-1}^{k-1} \end{bmatrix}.$$

We can prove that M_k^{-1} has entries bounded by 34 (independent of k).

The coefficients are too small, or negative

- Instead of working with N, we work with some $N' < N$ where $N - N'$ is a large positive integer linear combination of the $c_k(n), c_k(n+1), \ldots, c_k(n+k-1)$, with coefficients large enough to offset and negative coefficients in the representation of N'.
- We do this by a kind of greedy algorithm: we first define

$$d = \left\lfloor \frac{N}{c_k(n) + c_k(n+1) + \cdots + c_k(n+k-1)} \right\rfloor .$$

- Next we set $N_0 = N - d(c_k(n) + c_k(n+1) + \cdots + c_k(n+k-1))$ and successively set

$$N_i = N_{i-1} - c_k(n+k-i) \left\lfloor \frac{N_{i-1}}{c_k(n+k-i)} \right\rfloor .$$

  for $i = 1, 2, \ldots, k$.
- Then $N_k < c_k(n)$, and we work with N_k instead of N.

The coefficients are not integers

- We round the linear combination down to the nearest integer.
- This results in an error that is of the form $ac_k(n+i)/\Delta$, where Δ is the determinant of the Vandermonde matrix M_k.
- Each $c_k(n+i)$ is a sum of powers of 2, so it suffices to handle expressions of the form $2^j/\Delta$.
- To do this, we observe that $1/\Delta$ has (in base 2) a periodic expansion that consists of a block repeated infinitely often
- The length of the block depends on the order of 2 (mod Δ)
- We take enough copies of this block to get a k'th power.
- The remaining error is a constant depending only on k.

Open Problems

- Are there arbitrarily large even numbers that cannot be written as the sum of two binary palindromes? The sequence of unrepresentable numbers starts
  $$176, 188, 208, 242, 244, 310, 524, 628, 656, 736, 754, \ldots$$
- Does the set of numbers representable as the sum of two binary palindromes have positive density?
- Say something about the number of representations as the sum of two, three, or four palindromes.
- Are there arbitrarily large even numbers that cannot be written as the *difference* of two binary palindromes? The sequence of unrepresentable numbers starts
  $$1844, 1892, 2512, 3700, 4702, 5476, 5534, 7364, \ldots$$

Antipalindromes

An *antipalindrome* is a string of the form $x\overline{x^R}$, where $\overline{y}$ flips bits from 0 to 1 and vice versa.

Conjecture. Every even natural number except 8, 18, 28, 130, 134, 138, 148, 158, 176, 318, 530, 538, 548, 576, 644, 1300, 2170, 2202, 2212, 2228, 2230, 2248, 8706, 8938, 8948, 34970, 35082 is the sum of at most 3 antipalindromes.

Theorem. Every natural number is the sum of at most 7 generalized antipalindromes.