The Logical Approach to Automatic Sequences

Part 3: Proving Claims about Automatic Sequences with `Walnut`

Jeffrey Shallit
School of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
shallit@cs.uwaterloo.ca
https://cs.uwaterloo.ca/~shallit

Deciding properties of automatic sequences with `Walnut`

As we will see, the logical theory we discussed yesterday is powerful enough to express many assertions about automatic sequences.

Luckily, Hamoon Mousavi has created a Java prover `Walnut` that implements the decision procedure discussed yesterday, and it is publicly available.

There is also a software manual available that describes its use.

Today, we'll look at a variety of properties of automatic sequences and prove them using `Walnut`.

Ultimate periodicity

We can write a formula for ultimate periodicity of a sequence S as follows:

$$\exists n \geq 0 \; \exists p \geq 1 \; \forall j \geq n \; S[j] = S[j+p].$$

When we translate this to `Walnut`:

– we need not specify $n \geq 0$ explicitly, as this is implicit in the domain $\mathbb{N}$;

– we translate $\exists p \geq 1 \cdots$ to

```
Ep (p >= 1) & ...
```

– we translate $\forall j \geq n \cdots$ to

```
Aj (j >= n) => ...
```

Checking ultimate periodicity of the Thue-Morse sequence $\mathbf{t}$

```
% cd Walnut/bin
% java Main.prover
eval tmup "En Ep (p>=1) & Aj (j >= n) => T[j] = T[j+p]":
```

Now go and check the file `tmup.txt` in the directory `Walnut/Result`, and it says "false".

So the Thue-Morse sequence is not ultimately periodic.

Ultimate periodicity

Theorem. If a k-DFAO of n states generates an ultimately periodic sequence S, then the preperiod and period are bounded by $k^{3 \cdot 2^{4n^2}}$.

Proof. We can make a DFA accepting those $(j, l)_k$ such that $S[j] = S[l]$ in n^2 states. We can enforce $(j \geq n) \wedge (l = j + p)$ using a total of $4n^2$ states. Checking $\forall j$ requires some nondeterminism and another negation, giving 2^{4n^2}. Finally, checking $p \geq 1$ takes 3 states, so $3 \cdot 2^{4n^2}$ states. Such an automaton, if it accepts anything at all, must accept p and n having at most $3 \cdot 2^{4n^2}$ symbols.

Better results: Honkala, Sakarovitch, etc.

Squares

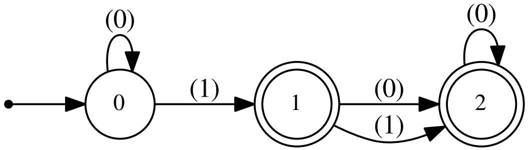
We can write a formula for the orders of squares in a sequence S as follows:

$$(n > 0) \ \wedge \exists i \ \forall j \ (j < n) \implies S[i+j] = S[i+j+n]$$

In `Walnut`, for the Thue-Morse sequence, this is done with the command

```
eval tmsq "(n>0) & Ei Aj (j<n) => T[i+j] = T[i+j+n]":
```

Then we go and look in the `Result` directory for `tmsq.gv`.



Thus there are squares of order 2^n and $3 \cdot 2^n$ for all $n \geq 0$ in the Thue-Morse sequence. Where are they?

Squares

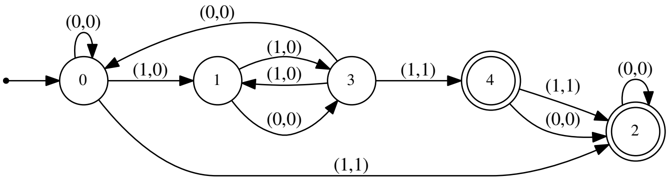
We can write a formula for the positions and orders of squares in a sequence S as follows:

$$(n > 0) \;\wedge \forall j \; (j < n) \implies S[i+j] = S[i+j+n]$$

In `Walnut`, for the Thue-Morse sequence, this is

```
eval tmsqp "(n>0) & Aj (j<n) => T[i+j] = T[i+j+n]":
```

Then we go and look in the `Result` directory for `tmsqp.gv`.



Overlaps

We can write a formula for the orders and positions of overlaps in a sequence S as follows:

$$(n \geq 1) \ \wedge \ \forall j \ (j \leq n) \implies S[i+j] = S[i+j+n]$$

When we do this in `Walnut` for the Thue-Morse sequence we type

```
eval tmover "(n>=1) & Aj (j<=n) => T[i+j] = T[i+j+n]":
```

which gives an automaton that accepts nothing.

Arbitrary fractional powers

Fractional powers are generalizations of integer powers.

We say a string x is a (ℓ/p)-power if it is of length ℓ and has period p.

For example, `ionization` is a $(10/7)$-power.

We say a word w avoids α powers, for $\alpha > 1$ a real number, if w has no factor that is a (ℓ/p)-power for $(\ell/p) \geq \alpha$.

We say a word w avoid α^+ powers if w has no factor that is a (ℓ/p)-power for $(\ell/p) > \alpha$.

Thus, avoiding squares is avoiding 2-powers, and avoiding overlaps is avoiding 2^+-powers.

Arbitrary fractional powers

We can write a formula for a word S avoiding α-powers:

$$\neg(\exists i\ \exists n\ (n \geq 1)\ \wedge\ \forall j\ (j+n < \alpha n) \implies S[i+j] = S[i+j+n])$$

or avoiding α^{+}-powers:

$$\neg(\exists i\ \exists n\ (n \geq 1)\ \wedge\ \forall j\ (j+n \leq \alpha n) \implies S[i+j] = S[i+j+n])$$

In order for this to be expressible in our logical theory, we must have $\alpha = \ell/p$ for some integers ℓ, p. Then we rewrite

$$j+n < \alpha n \quad \text{as} \quad \ell j < (p-\ell)n$$

and

$$j+n \leq \alpha n \quad \text{as} \quad \ell j \leq (p-\ell)n.$$

Arbitrary fractional powers

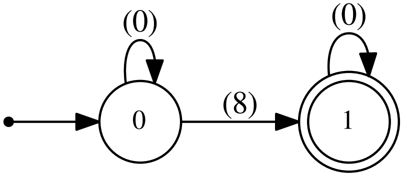
Example: the Leech sequence:

$0 \to 0121021201210; \quad 1 \to 1202102012021; \quad 2 \to 2010210120102$

This sequence avoids $(15/8)^+$ powers and has infinitely many $(15/8)$-powers. We can create a file named `LE.txt` in the `Word Automata` directory that implements this morphism. Then we say

```
eval le158 "?msd_13 Ei (n>=1) & Aj (8*j < 7*n) => LE[i+j] = LE[i+j+n]":
```

After a reasonable delay we get the automaton



which says that there are powers $x^{15/8}$ for $|x| = 8 \cdot 13^i$ and $i \geq 0$.

Antisquares

Antisquares are binary words of the form $x\overline{x}$, where $\overline{x}$ means change 0 to 1 and 1 to 0.

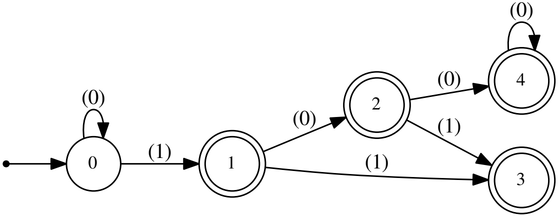
A formula for lengths of antisquares:

$$Ei\ (n \geq 1)\ \wedge\ \forall j\ (j < n)\ \implies\ S[i+j] \neq S[i+j+n].$$

Antisquares

Let's compute antisquare orders for the Rudin-Shapiro sequence in `Walnut`:

```
Ei (n>=1) & Aj (j<n) => RS[i+j]!=RS[i+j+n]
```



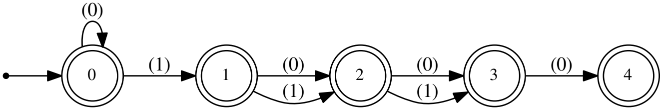
This gives the following orders of antisquares: 2^i for $i \geq 0$ and 3 and 5.

Palindromes

We can write a formula for the positions of palindromes in a sequence S as follows:

$$\exists i\ \forall j\ (j < n) \implies S[i+j] = S[(i+n)-(j+1)]$$

When we do this for the Rudin-Shapiro sequence we get



So the only palindrome lengths in Rudin-Shapiro are

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14\}.$$

Maximal palindromes

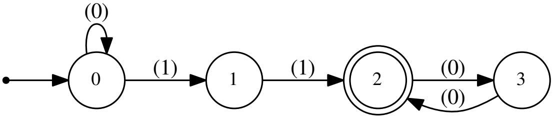
A *maximal palindrome* in a word S is a palindrome x such that axa does not appear in S for any a.

Formula:

$$\exists i\ (\forall j\ (j < n) \implies S[i+j] = S[(i+n)-(j+1)]) \wedge (\forall l\ (((l > 0) \wedge (\forall\ m\ (m < n) \implies S[l+m] = S[i+m])) \implies (S[l-1] \neq S[l+n])))$$

Maximal palindromes

When we do this for the Thue-Morse sequence, we get only the lengths $3 \cdot 4^i$ for $i \geq 0$.



Exercise: How could you use `Walnut` to prove that the only maximal palindromes in the Thue-Morse sequence are $\mu^{2n}(010)$ and $\mu^{2n}(101)$ for $n \geq 0$?

Reversal-freeness

Formula:

$$\forall i \; \forall j \; \exists k \; (k < n) \; \wedge \; S[i+k] \neq S[(j+n)-(k+1)].$$

This says that for any word of length n beginning at position i, all other words beginning at all positions j have their reversal $S[j..j+n-1]$ differing at some position k from $S[i..i+n-1]$.

Here is an example: take the morphism defined by

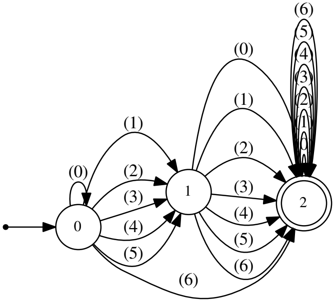
$$0 \rightarrow 0001011; \qquad 1 \rightarrow 0010111.$$

This gives a 7-automatic sequence. We encode this in a file `RSR.txt`. Then we use the `Walnut` command:

Reversal-freeness

```
?msd_7 Ai Aj Ek (k<n) & RSR[i+k] != RSR[(j+n)-(k+1)]
```

and we get the automaton below. So the only lengths for which words and their reversals are both present are $0, 1, 2, 3, 4, 5$.



Recurrence

Recall: **x** is recurrent if every factor that occurs, occurs infinitely often.

Equivalent to: for every factor that occurs, there is another occurrence at a higher index. Formula

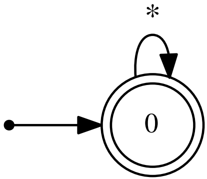
$$\forall i\ \forall n\ \exists j\ (j > i) \wedge (\forall l (l < n) \implies S[i+l] = S[j+l]).$$

Recurrence

When we run

```
Ai An Ej ((j>i) & (Al (l<n) => T[i+l] = T[j+l]))
```

in `Walnut` for the Thue-Morse sequence we get the automaton



which is `Walnut`'s way to represent an automaton that accepts everything with 0 free variables.

Bordered and unbordered factors

A nonempty word x is bordered if there is a nonempty word w and a possibly empty word t such that $x = wtw$. For example, `ionization` is bordered with border `ion`.

Formula for $S[i..i+n-1]$ being bordered:

$$\exists l \ (0 < l) \wedge (l < n) \wedge (\forall j \ (j < l) \implies S[i+j] = S[(i+n+j)-l])$$

In `Walnut` we can define a macro for this:

```
def tmbord "El (0<l) & (l<n) & (Aj (j<l) => T[i+j]=T[(i+n+j)-l] )":
```

and then use it by saying

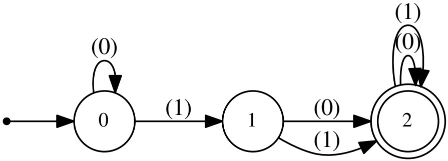
```
eval tmborders "Ei $tmbord(i,n)":
```

or

```
eval tmunbord "Ei ~$tmbord(i,n)":
```

Bordered factors

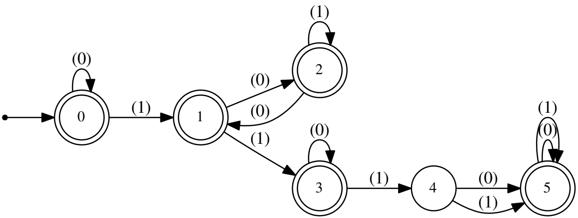
When we do this for Thue-Morse we get



for the lengths of bordered factors of Thue-Morse, which shows that there is a bordered factor for all lengths > 1.

Unbordered factors

When we do this for unbordered factors we get



for the lengths of unbordered factors of Thue-Morse.

So we have proved: there is an unbordered factor of length n of the Thue-Morse sequence iff $(n)_2 \not\in 1(01^*0)^*10^*1$. This improves a 2009 result due to Currie and Saari; they proved $\mathbf{t}$ has an unbordered factor of length n if $n \not\equiv 1 \pmod 6$.

Balanced words

A word x is *balanced* if $||y|_a - |z|_a| \leq 1$ for all equal-length factors y, z of x and all letters a.

It is not clear how to state this in first-order logic.

Luckily there is an alternative characterization, which is often quoted as

x is unbalanced iff there exists a palindrome p such that both $0p0$ and $1p1$ are both factors of x .

But an even simpler characterization is

x is unbalanced iff there exists a word y (not necessarily a palindrome) such that both $0y0$ and $1y1$ are both factors of x.

These two characterizations are easily seen to be equivalent.

Balanced words

Here is a formula for unbalanced factors of length n:

$$(n \geq 2) \wedge \exists i \; \exists j \; (\forall k((0 < k) \wedge (k + 1 < n)) \implies (S[i+k] = S[j+k])) \wedge S[i] = 0 \;\wedge\; S[j] = 1 \;\wedge\; S[i+n-1] = 0 \;\wedge\; S[j+n-1] = 1$$

In `Walnut` this is

```
(n>=2) & Ei Ej (Ak ((0<k)&(k+1<n)) => S[i+k] = S[j+k])
 & S[i]=@0 & S[i+n-1] = @0 & S[j] = @1 & S[j+n-1] = @1
```

Rich words

We can count the number of distinct palindromes occurring in a word.

For example, the word `Mississippi` has the following distinct nonempty palindromes in it:

$$\mathtt{M}, \mathtt{i}, \mathtt{s}, \mathtt{p}, \mathtt{ss}, \mathtt{pp}, \mathtt{sis}, \mathtt{issi}, \mathtt{ippi}, \mathtt{ssiss}, \mathtt{ississi}$$

Theorem. Every word of length n contains, as factors, at most n distinct palindromes.

Proof. For each index p of a word w, consider the palindromes ending at this index. Suppose at least two palindromes, x and y occur for the first time ending at p. Then wlog $|x| < |y|$. So then x is a suffix of y, so $x^R = x$ is a prefix of y, contradicting the claim that x occurred for the first time ending at p.

So at each position p at most 1 new palindrome can end.

Rich words

We say that a length-n word is `rich` if it contains, as factors, exactly n distinct nonempty palindromes.

We can therefore make a formula for the factor $S[i..i+n-1]$ being rich as follows: at each position p there is a palindrome ending at p that doesn't occur earlier in that factor.

Exercise. Write a predicate for richness and test it on the Thue-Morse sequence. You should find that there are no rich factors of length > 16.

Exercise. Find a 2-automatic sequence where all factors are rich, and prove it using `Walnut`.

Primitive words

A nonempty word w is *primitive* if it cannot be written as x^e with $e \geq 2$. So a primitive word is a non-power.

It's easy to see that a word w is a nontrivial power if and only if there is some cyclic shift (by $0 < j < |w|$ positions) of w that is equal to w. So we can write a formula for $S[i..i+n-1]$ being a power as follows:

$$\exists j,\ 0 < j < n,\ ((\forall t < n-j\ S[i+t] = S[i+j+t]) \wedge (\forall u < j\ S[i+u] = S[i+n+u-j]))$$

A formula for being primitive is just the negation of this.

The "substitute variables" trick

Recall our formula for primitivity:

$$\neg\exists j,\ 0 < j < n,\ ((\forall t < n - j\ S[i+t] = S[i+j+t]) \wedge (\forall u < j\ S[i+u] = S[i+n+u-j]))$$

This formula is correct, but indexing the automatic sequence by four variables (as in $i+n+u-j$) could be prohibitively expensive for our algorithm when the underlying automaton has many states.

To reduce the running time, use the substitution of variables $t' = i + t$ and $u' = i + u + n$ to get

$$\neg\exists j,\ 0 < j < n,\ ((\forall t',\ i \le t' < n + i - j,\ S[t'] = S[t' + j]) \wedge (\forall u', n + i \le u' < n + i + j,\ S[u' - n] = S[u' - j]))$$

This one is about twice as fast for the Thue-Morse sequence.

Privileged words

A word x is *privileged* if is of length ≤ 1, or it has a border w with $|x|_w = 2$ that is itself privileged. For example, `abracadabra` has a border `abra` that appears only at the beginning and end. And `abra` has a border `a` that occurs only at the beginning and end. Finally, `a` is privileged, and so is `abra` and so is `abracadabra`.

As stated it is not obvious that we can state this property in first-order logic.

However, there is another way to state the property (due to Luke Schaeffer): a word is privileged if for all n with $1 \leq n < |w|$ there exists a word x of length $\leq n$ such x is a border of w and there is exactly one occurrence of x in the first n symbols of w and one occurrence of x in the last n symbols of w.

Exercise: write a predicate for the privileged property, and run it on the Thue-Morse word.

Closed words

A word x is called `closed` if it is of length ≤ 1, or if it has a border w with $|x|_w = 2$.

For example, `alfalfa` is a closed word because of the border `alfa`. On the other hand, although `academia` is bordered, it is not closed.

Theorem. There is a closed factor of the Thue-Morse word **t** of every length.

Common factors

Arbitrarily large common factors between two k-automatic sequences:

$$\exists i\ \exists j\ \forall k\ (k < n) \implies R[i+k] = S[j+k]$$

If two k-automatic sequences, generated by automata of s and t states, respectively, have a factor of length $\ell >$ ... in common, then they have arbitrarily long factors in common.

Automatic reals and Lehr's proof of closure under addition

A real number x is said to be k-automatic in base-b if its base-b expansion mod 1 is generated by a k-DFAO. The set of all such numbers is written $L(k, b)$.

Example: the Thue-Morse real number

$$0.0110100110010110 \cdots$$

is 2-automatic in base 10.

Exercise: how can we show that $L(k, b)$ forms a $\mathbb{Q}$-vector space? The difficulty comes because carries can come from arbitrarily far to the right.

Other decidable things: critical exponents

- The *critical exponent* of a word **w** is the supremum, over all factors x of **w**, of the exponent of x.
- The critical exponent of the Thue-Morse word **t** is 2.

Representing rational numbers

- Represent rational number $\alpha = p/q$ by pair of integers (p, q), represented in base k; pad shorter with leading zeroes
- So representations of rationals are over the alphabet $\Sigma_k \times \Sigma_k$
- For example, if $w = [3, 0][5, 0][2, 4][6, 1]$ then $[w]_{10} = (3526, 41)$.
- Define $\text{quo}_k(x) = [\pi_1(x)]_k/[\pi_2(x)]_k$, where π_i is the projection onto the i'th coordinate
- So $\text{quo}_{10}(w) = 3526/41 = 86$.
- Canonical representations lack leading $[0, 0]$'s
- Every rational has infinitely many canonical representations, e.g., as $(1, 2), (2, 4), (3, 6), \ldots$, etc.

Automatic sets over $\mathbb{Q}^{\geq 0}$

- $\text{quo}_k(L) = \bigcup_{x \in L} \{\text{quo}_k(x)\}$

- $A \subseteq \mathbb{Q}^{\geq 0}$ is a k-automatic set of rationals if $A = \text{quo}_k(L)$ for some regular language $L \subseteq (\Sigma_k \times \Sigma_k)^*$.
- *not* the same notion as the automatic reals of Boigelot, Brusten, and Bruyère

Examples

Example 1. Let $k = 2$, $B = \{[0,0],[0,1],[1,0],[1,1]\}$, and consider

$$L_1 := B^*\{[0,1],[1,1]\}B^*.$$

Then L_1 consists of all pairs of integers where the second component has at least one nonzero digit — the point being to avoid division by 0. Then $\text{quo}_2(L) = \mathbb{Q}^{\geq 0}$, the set of all non-negative rational numbers.

Example 2. Consider

$$L_2 = \{w \in (\Sigma_k^2)^* \ : \ \pi_1(w) \in 0^* C_k \text{ and } \pi_2(w) \in 0^*1\}.$$

Then $\text{quo}_2(L_2) = \mathbb{N}$.

Example 3. Let $k = 3$, and consider the language

$$L_3 := [0,1]\{[0,0],[2,0]\}^*.$$

Then $\mathrm{quo}_3(L_3)$ is the 3-*adic Cantor set*, the set of all rational numbers in the "middle-thirds" Cantor set with denominators a power of 3.

Example 4. Let $k = 2$, and consider

$$L_4 := [0,1]\{[0,0],[0,1]\}^*\{[1,0],[1,1]\}.$$

Then the numerator encodes the integer 1, while the denominator encodes all positive integers that start with 1. Hence

$$\mathrm{quo}_2(L_4) = \{\frac{1}{n} \,:\, n \geq 1\}.$$

Example 5. Let $k = 4$, and consider

$$S := \{0, 1, 3, 4, 5, 11, 12, 13, \ldots\}$$

of all non-negative integers that can be represented using only the digits $0, 1, -1$ in base 4. Consider the language

$$L_5 = \{(p, q)_4 \, : \, p, q \in S\}.$$

It is not hard to see that L_5 is $(\mathbb{Q}, 4)$-automatic.
The main result in Loxton & van der Poorten [1987] can be rephrased as follows: $\text{quo}_4(L_5)$ contains every odd integer.
In fact, an integer t is in $\text{quo}_4(L_5)$ if and only if the exponent of the largest power of 2 dividing t is even.

Example 6. Consider

$$L_6 = \{w \in (\Sigma_k^2)^* \ : \ \pi_2(w) \in 0^*1^+0^*\}.$$

An easy exercise using the Fermat-Euler theorem shows that that $\mathrm{quo}_2(L_6) = \mathbb{Q}^{\geq 0}$.

Examples

Example 7. For a word x and letter a let $|x|_a$ denote the number of occurrences of a in x. Consider the regular language

$$L_7 = \{w \in (\Sigma_2^2) \ : \ |\pi_1(w)|_1 \text{ is even and } |\pi_2(w)|_1 \text{ is odd}\}.$$

Then it follows from a result of Schmid [1984] that

$$\mathrm{quo}_2(L_7) = \mathbb{Q}^{\geq 0} - \{2^n \ : \ n \in \mathbb{Z}\}.$$

Basic decidability properties

Given a DFA M accepting a language L representing a set of rationals S, can decide

- if $S = \emptyset$
- given $\alpha \in \mathbb{Q}^{\geq 0}$, whether there exists $x \in S$ with $x = \alpha$ (resp., $x < \alpha$, $x \leq \alpha$, $x > \alpha$, $x \geq \alpha$, $x \neq \alpha$, etc.)
- if $|S| = \infty$
- given a finite set $F \subseteq \mathbb{Q}^{\geq 0}$, if $F \subseteq S$ or if $S \subseteq F$
- given $\alpha \in \mathbb{Q}^{\geq 0}$, if α is an accumulation point of S

sup A is rational or infinite

Given a DFA M accepting $L \subseteq (\Sigma_k \times \Sigma_k)^*$ representing a set of rationals $A \subseteq \mathbb{Q}^{\geq 0}$, what can we say about $\sup A$?

Theorem. $\sup A$ is rational or infinite, and is computable.

Proof ideas: $\text{quo}_k(uv^i w)$ forms a monotonic sequence. Defining

$$\gamma(u, v) := \frac{[\pi_1(uv)]_k - [\pi_1(u)]_k}{[\pi_2(uv)]_k - [\pi_2(u)]_k}$$

one of the following three cases must hold:

(i) $\text{quo}_k(uw) < \text{quo}_k(uvw) < \text{quo}_k(uv^2 w) < \cdots < U$;
(ii) $\text{quo}_k(uw) = \text{quo}_k(uvw) = \text{quo}_k(uv^2 w) = \cdots = U$;
(iii) $\text{quo}_k(uw) > \text{quo}_k(uvw) > \text{quo}_k(uv^2 w) > \cdots > U$.

Furthermore, $\lim_{i \to \infty} \text{quo}_k(uv^i w) = U$.

sup A is rational or infinite

It follows that if $\sup A$ is finite, and the DFA M has n states, then $\sup A = \max T$, where

$$T = T_1 \cup T_2$$

and

$$\begin{aligned} T_1 &= \{\mathrm{quo}_k(x) \ : \ |x| < n \text{ and } x \in L\}; \\ T_2 &= \{\gamma(u,v) \ : \ |uv| \leq n, \ |v| \geq 1, \ \delta(q_0,u) = \delta(q_0,uv), \\ &\qquad \text{and there exists } w \text{ such that } uvw \in L\}. \end{aligned}$$

sup A is computable

We know that $\sup A$ lies in the finite computable set T.

For each of $t \in T$, we can check to see if $t \geq \sup A$ by checking if $A \cap (t, \infty)$ is empty.

Then $\sup A$ is the least such t.

Computing the critical exponent

- Previously known to be computable for fixed points of uniform morphisms (Krieger)

Theorem. If **w** is a k-automatic sequence, then its critical exponent is rational or infinite. Furthermore, it is computable from the DFAO M generating w.

Proof sketch. Given M, we can transform it into another automaton M' accepting

$$\{(m, n) : \text{there exists } i \geq 0 \text{ such that } \mathbf{w}[i..i+m-1] \text{ has period } n\}.$$

We then apply our algorithm for computing $\sup(\text{quo}_k(L))$ to $L(M')$.

Open Questions

- Extend these ideas to morphic sequences (fixed points of possibly non-uniform morphisms, followed by a coding)
  - Some ideas are extendable to, e.g., the Fibonacci word
  - Carton & Thomas proved that $(\mathbb{N}, <, \text{morphic word})$ is decidable
- Which predicates for automatic sequences (like squarefreeness) are decidable in polynomial time? Leroux has proved it for ultimate periodicity.

More open problems

- Extend these ideas to "infinite state" automata (i.e., fixed points of morphism like $n \to (an + b, cn + d)$) or prove undecidability
- Is $\sup\{x/y : (x, y)_k \in L\}$ computable for context-free languages L?
- Given a regular language $L \subseteq (\Sigma_k \times \Sigma_k)^*$ representing a set $S \subseteq \mathbb{N} \times \mathbb{N}$ of pairs of natural numbers, is it decidable if S contains a pair (p, q) with $p \mid q$?
  - This is a question of $\exists^1(\mathbb{N}, +, V_k, |)$; of course $\text{Th}(\mathbb{N}, +, |)$ is undecidable and $\exists^1(\mathbb{N}, +, |)$ is decidable (Lipshitz)

More Open Questions

- Prove or disprove: if L is a regular language with $\text{quo}_k(L) = \mathbb{Q}^{\geq 0}$, then L contains infinitely many distinct representations for infinitely many distinct rational numbers.

Which of the following questions is decidable? Given L representing a set of rationals S,

- Is there some rational $p/q \in S$ having infinitely many distinct representations in L?
- Are there infinitely many distinct rationals $p/q \in S$ having infinitely many distinct representations in L?