Formal Languages and Number Theory

Jeffrey Shallit
Department of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
shallit@graceland.uwaterloo.ca
http://www.math.uwaterloo.ca/~shallit

Outline of the Talk

This talk is about application of number theory to problems of formal language theory and vice versa.

Table of Contents

1. State complexity of the intersection of unary languages.
2. Descriptional complexity of unary context-free grammars representing regular languages.
3. Primitive words and context-free languages
4. Automatic complexity.

Much of this is joint work with my graduate student, Ming-wei Wang.

State Complexity

The *state complexity* of a regular language L, $\mathrm{sc}(L)$, is the minimum number of states needed to accept it by a DFA.

The problem:
Given languages L, L' with state complexity n, n' respectively, what are good bounds on the state complexity of $L \cup L'$, LL', L^*, etc.?

For the state complexity of intersection, we have the following bound:

Proposition. *We have*

$$\mathrm{sc}(L \cap L') \leq \mathrm{sc}(L)\mathrm{sc}(L').$$

Proof. Use the usual direct product construction.

State Complexity of Intersection

The upper bound of $\mathrm{sc}(L)\mathrm{sc}(L')$ can be achieved if L, L' are over an alphabet of size at least 2:

Proposition. $(\mathrm{S.\ Yu.})$ *Define*

$$L := \{x \in \{a,b\}^* \ : \ |x|_a \equiv 0 \pmod{n}\};$$

$$L' := \{y \in \{a,b\}^* \ : \ |y|_b \equiv 0 \pmod{n'}\}.$$

Then

$$\mathrm{sc}(L \cap L') = nn'.$$

But what if L, L' are *unary*, that is, defined over an alphabet of one symbol?

Clearly if $\gcd(n, n') = 1$ then the bound nn' can again be achieved, by taking $L = (a^n)^*$ and $L' = (a^{n'})^*$.

But what if $\gcd(n, n') > 1$?

State Complexity of Intersection for Unary Languages

A connected unary DFA has the property that its transition diagram consists of

- a *tail* of $t \geq 0$ states and
- a *cycle* of $c \geq 1$ states.

It is then not hard to prove that

Theorem. *Let M, M' be unary DFA's with tails of size t, t' and cycles of size c, c', respectively. If L, L' are the corresponding languages, we have*

$$\mathrm{sc}(L \cap L') \leq \max(t, t') + \mathrm{lcm}(c, c'). \qquad (1)$$

Furthermore, for all $t, t' \geq 0$ and $c, c' \geq 1$ there exist unary languages for which the bound (1) is achieved.

For example, if $t \geq t'$, take

$$\begin{aligned} L &= a^{t+c-1}(a^c)^*; \\ L' &= a^r(a^{c'})^*; \\ r &= t - 1 \bmod c'. \end{aligned}$$

Two New Number-Theoretic Functions

Thus, to estimate the worst-case behavior for the state complexity of intersection of unary languages with n and n' states, respectively, we must estimate the function

$$F(n,n') = \max_{\substack{1\le c\le n\\ 1\le c'\le n'}} \left(\max(n-c, n'-c') + \operatorname{lcm}(c,c')\right).$$

This in turn suggests studying the somewhat simpler and more natural function

$$G(n,n') = \max_{\substack{1\le c\le n\\ 1\le c'\le n'}} \operatorname{lcm}(c,c').$$

- The asymptotic behavior of F and G is still not known precisely
- There is a relation to JACOBSTHAL'S function $g(n)$, which is the *least integer r such that every set of r consecutive integers contains at least one integer relatively prime to n.*
- IWANIEC proved [1978] using the linear sieve that $g(n) = O((\log n)^2)$.
- It then follows that if $n \le n'$, we have $F(n,n') \ge G(n,n') \ge nn' - c(\log n)^2 n$ for some constant c.

Descriptional Complexity of Context-Free Grammars

- Can measure the size of a context-free grammar as the number of symbols needed to write down its description.
- Suppose a CFG G generates a regular language. How big can the corresponding DFA be, in terms of the size of G?
  - If the CFG is over an alphabet with at least 2 symbols, the answer is, there is no recursive bound.
  - More precisely, MEYER and FISCHER proved [1971] that given any recursive function f, for arbitrarily large integers n there exists a CFG of size n describing a regular language L such that any DFA accepting L has at least $f(n)$ states.
  - But how about the unary case?
  - It is possible to show that there exists a constant such that any unary CFG of size n describing a regular language can be accepted by a DFA with at most $O(2^{cn^2})$ states.
  - But is this bound achievable?

An Example Exhibiting 2^{cn^2} Blowup

$$
\begin{aligned}
A_0 &\rightarrow a \\
A_{i+1} &\rightarrow A_i A_i \quad (i \geq 0)
\end{aligned}
$$

$$\text{so } A_i \Longrightarrow^* \{a^{2^i}\}$$

$$B_i \rightarrow aA_i$$

$$\text{so } B_i \Longrightarrow^* \{a^{2^i+1}\}$$

$$
\begin{aligned}
C_0 &\rightarrow a \\
C_{i+1} &\rightarrow a \mid C_i C_i \quad (i \geq 0)
\end{aligned}
$$

$$\text{so } C_i \Longrightarrow^* \{a, a^2, a^3, \ldots, a^{2^i}\}$$

$$D_i \rightarrow D_i B_i \mid C_i \quad (i \geq 0)$$

$$
\begin{aligned}
\text{so } D_i \Longrightarrow^* & \{a, a^2, a^3, \ldots, a^{2^i}\}\{a^{2^i+1}\}^* \\
= & \{a^j \ : \ j \not\equiv 0 \pmod{2^i+1}\}.
\end{aligned}
$$

An Example Exhibiting 2^{cn^2} Blowup

And finally, let

$$S_i \quad \rightarrow \quad \epsilon \mid D_0 \mid D_1 \mid D_2 \mid \cdots \mid D_i$$

so $S_i \stackrel{*}{\Longrightarrow} \{\epsilon\} \cup$
$\{a^k \ : \ k \not\equiv 0 \pmod{\text{lcm}(2^0+1, 2^1+1, \ldots, 2^i+1)}\}.$

Now let $G_n = (V_n, \{a\}, P_n, S_n)$, where

$$V_n = \{A_i, B_i, C_i, D_i, S_i \ : \ 0 \leq i \leq n\}$$

and P_n is the set of $O(n)$ productions given above involving these variables.

It is clear that $L(G_n)$ is regular.

The shortest string not generated by G_n is of length

$$\text{lcm}(2^0+1, 2^1+1, \ldots, 2^n+1)$$

and so any DFA accepting $L(G_n)$ must have at least this many states.

It remains to estimate

$$\text{lcm}(2^0+1, 2^1+1, \ldots, 2^n+1)$$

An Example Exhibiting 2^{cn^2} Blowup

We use the following theorem of Bézivin [1989]:

Theorem. *Let a, b be integers with $b \neq 0$ and $\gcd(a, b) = 1$. Let α, β be zeroes of the polynomial $X^2 - aX - b$. For $m \geq 2$ define*

$$u_m(n) = \frac{\alpha^{mn} - \beta^{mn}}{\alpha^n - \beta^n}.$$

Then

$$\lim_{n\to\infty} \frac{\log(u_m(1)u_m(2)\cdots u_m(n))}{\log \operatorname{lcm}(u_m(1), u_m(2), \ldots, u_m(n))} = \frac{(m-1)L(m)\pi^2}{6H(m)},$$

where

$$L(m) = \prod_{p \,|\, m} \left(1 - \frac{1}{p^2}\right)$$

and

$$H(m) = \sum_{\substack{d \,|\, m \\ d>1}} \frac{\varphi(d)\varphi(m/d)d}{m}.$$

Now take $a = 3, b = -2, m = 2$. Then $\alpha = 2$ and $\beta = 1$, and we obtain

$$\lim_{n\to\infty} \frac{\log((2^0+1)(2^1+1)\cdots(2^n+1))}{\log \operatorname{lcm}(2^0+1, 2^1+1, \ldots, 2^n+1))} = \frac{\pi^2}{8}.$$

An Example Exhibiting 2^{cn^2} Blowup

On the other hand, it is easy to see that

$$\lim_{n\to\infty} \frac{(2^0+1)(2^1+1)\cdots(2^n+1)}{2^0 \cdot 2^1 \cdots 2^n} = c_1,$$

where $c_1 \doteq 4.768$, so it follows that

$$\log((2^0+1)(2^1+1)\cdots(2^n+1)) \sim \log c_1 + \frac{n(n+1)}{2}\log 2.$$

Putting this together with the Bézivin result, we get

$$\log \operatorname{lcm}(2^0+1, 2^1+1, \ldots, 2^n+1) \sim \frac{4\log 2}{\pi^2} n^2.$$

The Primitive Words Problem

- Let Σ be a finite alphabet with at least two letters.
- A word $w \in \Sigma^*$ is said to be *primitive* if it cannot be expressed in the form x^k with $k \geq 2$.
- Open problem in formal languages: is the language P of primitive words context-free?
- Answer is almost certainly no, but nobody knows how to prove this.
- PETERSEN [1996] proved the weaker result that P is not unambiguously context-free.
- He did this using the CHOMSKY-SCHÜTZENBERGER theorem, which states that if L is a context-free language having an unambiguous grammar, and $a_n := |L \cap \Sigma^n|$, then $\sum_{n \geq 0} a_n X^n$ is a formal power series in $\mathbb{Z}[[X]]$ which is algebraic over $\mathbb{Q}(X)$.
- Recently a remarkably simple proof of Petersen's result was found by ALLOUCHE using the theory of automatic sequences.

Example of Chomsky-Schützenberger Theorem

Consider the unambiguous grammar

$$\begin{aligned} S &\rightarrow M \mid U \\ M &\rightarrow 0M1M \mid \epsilon \\ U &\rightarrow 0S \mid 0M1U \end{aligned}$$

which represents strings of "if-then-else" clauses.

Then this has the following commutative image:

$$\begin{aligned} S &= M + U \\ M &= x^2M^2 + 1 \\ U &= Sx + x^2MU \end{aligned}$$

This system has the following power series solutions:

$$\begin{aligned} M &= 1 + x^2 + 2x^4 + 5x^6 + 14x^8 + 42x^{10} + \cdots \\ U &= x + x^2 + 3x^3 + 4x^4 + 10x^5 + 15x^6 + 35x^7 + 56x^8 + \cdots \\ S &= 1 + x + 2x^2 + 3x^3 + 6x^4 + 10x^5 + 20x^6 + 35x^7 + \cdots \end{aligned}$$

By the Chomsky-Schützenberger theorem, each variable satisfies an algebraic equation over $\mathbb{Q}(x)$.

For example, we have

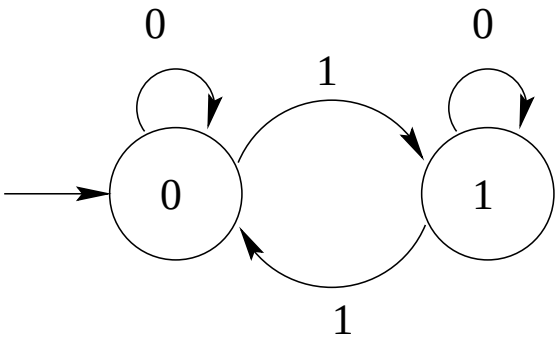
$$x(2x-1)S^2 + (2x-1)S + 1 = 0$$

Automata as Computers of Sequences

- We can generalize our notion of automaton to provide an output, not simply accept/reject.
- Formally, we define a *deterministic finite automaton with output* (DFAO) as a sextuple: $(Q, \Sigma, \delta, q_0, \Delta, \tau)$, where Δ is the finite *output alphabet* and $\tau : Q \to \Delta$ is the *output mapping*.
- Next, we decide on a integer base $k \geq 2$ and represent n as a string of symbols over the alphabet $\Sigma = \{0, 1, 2, \ldots, k-1\}$.
- To compute f_n, given an automaton M, express n in base-k, say, $a_r a_{r-1} \cdots a_1 a_0$, and compute $f_n = \tau(\delta(q_0, a_r a_{r-1} \cdots a_1 a_0))$.
- Any sequence that can be computed in this way is said to be k-automatic.

Example: The Thue-Morse sequence

- The THUE-MORSE sequence $(t_n)_{n\geq 0}$ is defined as follows: t_n is the parity of the number of 1's in the binary expansion of n.
- $(t_n)_{n\geq 0} = 0\,1\,1\,0\,1\,0\,0\,1\,\cdots$
- We have $t_0 = 0$; $t_{2n} = t_n$, and $t_{2n+1} = 1 - t_n$ for $n \geq 0$.
- THUE (c. 1906) studied this sequence because it is *cubefree*: it contains no subword of the form www, where w is a nonempty word.
- It is computed by the following DFAO:



Robustness of the Notion of Automatic Sequence

- the order in which the base-k digits are fed into the automaton in does not matter (provided it is fixed for all n);
- other representations also work (such as expansion in base-$(-k)$);
- automatic sequences are closed under many operations, such as shift, periodic deletion, q-block compression, and q-block substitution.
- if a symbol in an automatic sequence occurs with well-defined frequency r, then r is rational.

The Theorem of Christol

Theorem. (Christol [1980]). Let $(u_n)_{n\geq 0}$ be a sequence over

$$\Sigma = \{0, 1, \ldots, p-1\},$$

where p is a prime. Then the formal power series $U(X) = \sum_{n\geq 0} u_n X^n$ is algebraic over $GF(p)[X]$ if and only if $(u_n)_{n\geq 0}$ is p-automatic.

Example.

Let, as before, $(t_n)_{n\geq 0}$ denote the Thue-Morse sequence, i.e., $t_n =$ sum of the bits in the binary expansion of n, mod 2. Then $t_{2n} \equiv t_n$ and $t_{2n+1} \equiv t_n + 1$. If we set $A(X) = \sum_{n\geq 0} t_n X^n$, then

$$\begin{aligned} A(X) &= \sum_{n\geq 0} t_{2n} X^{2n} + \sum_{n\geq 0} t_{2n+1} X^{2n+1} \\ &= \sum_{n\geq 0} t_n X^{2n} + X \sum_{n\geq 0} t_n X^{2n} + X \sum_{n\geq 0} X^{2n} \\ &= A(X^2) + X A(X^2) + X/(1-X^2) \\ &= A(X)^2(1+X) + X/(1+X)^2. \end{aligned}$$

Hence $(1+X)^3 A^2 + (1+X)^2 A + X = 0$.

Back to Primitive Words

- Let $\psi_k(n)$ be the number of primitive words of length n over a k-letter alphabet.
- Then it is easy to see (using MÖBIUS inversion) that

$$\psi_k(n) = \sum_{d \mid n} \mu(d) k^{n/d}.$$

- If P_k were unambiguously context-free then by the CHOMSKY-SCHÜTZENBERGER theorem

$$R(X) = \sum_{n \geq 1} \psi_k(n) X^n$$

would be algebraic over $\mathbb{Q}(X)$.

- Then

$$R'(X) = \sum_{n \geq 1} \frac{\psi_k(n)}{k} X^n$$

would also be algebraic over $\mathbb{Q}(X)$.

- Let p be a prime dividing k. Then it is not hard to see that

$$R'_p(X) = \sum_{n \geq 1} (\frac{\psi_k(n)}{k} \bmod p) X^n$$

would also be algebraic over $GF(p)(X)$.

- But

$$\begin{aligned}\frac{\psi_k(n)}{k} &= \sum_{d \mid n} \mu(d) k^{n/d-1} \\ &= \mu(n) + \sum_{\substack{d \mid n \\ d \neq n}} \mu(d) k^{n/d-1} \\ &\equiv \mu(n) \pmod{p}.\end{aligned}$$

- It follows that

$$R'_p(X) = \sum_{n \geq 1} \mu(n) X^n$$

  and so the sequence $(\mu(n) \bmod p)_{n \geq 0}$ must be p-automatic.

- But then $(\mu(n)^2 \bmod p)_{n \geq 0}$ would be p-automatic.

- However, $\mu(n)^2 \equiv 1 \pmod{p}$ if and only if n is squarefree.

- By a classical theorem, the density of the squarefree numbers exists and is equal to $6/\pi^2$, an irrational number.

- But the density of symbols in automatic sequences (if it exists) must be rational, a contradiction.

- It follows that $R(X)$ is not algebraic over $\mathbb{Q}(X)$ and so P_k is not unambiguously context-free.

Automatic Complexity: Introduction

- We're interested in a measure of complexity for finite strings x over a finite alphabet, typically $\{0, 1\}$.
- Any such measure should reflect, in some sense, how "complicated" the string x is.
- Old idea: KOLMOGOROV-CHAITIN complexity
  - $K(x) =$ size of the shortest pair (M, y) where M is a Turing machine description and y is an input, such that M outputs x on input y.
  - Extremely useful; see book by LI and VITÁNYI.
  - The map $x \to (M, y)$ can be viewed as the best possible way to compress x.
  - Unfortunately not computable; no effective procedure for finding the pair (M, y).
  - Also depends on the particular model of universal TM chosen.
- Open problem: is it true that $K(xx) > K(x)$ for almost all strings x? Note: it is easy to see that $K(xx) = K(x) + O(1)$.

Alternatives to Kolmogorov Complexity

- Can we find a computable measure of complexity?
- Idea: replace Turing machine with a less powerful model.
- Example: replace the Turing machine with a context-free grammar (CFG).
- We choose some measure of the complexity of a context-free grammar, and then ask for the "smallest" grammar G such that $L(G) = \{x\}$.
- If we demand that the CFG be in Chomsky normal form (i.e., all productions are of the form $A \to BC$ or $A \to a$ where A, B, C are variables and a is a terminal), then we get a well-known measure of complexity associated with "word chains".
- Studied by Diwan; Berstel & Brlek; Roth; Arnold & Brlek, Althöfer, etc.
- Open problem: what is the complexity of computing a minimum word chain or its length?
  - Probably a difficult problem to study, since in the unary case we get the well-studied area of "addition chains", which are discussed in Knuth, Volume II.

Automatic Complexity

- Alternate idea: consider replacing the Turing machine with a deterministic finite automaton (DFA).
- First attempt: find a smallest DFA M such that $L(M) = \{x\}$.
- Uninteresting: if $|x| = n$, a smallest such DFA always has exactly $n + 2$ states.
- If a DFA M accepts a string x, but no other strings of length $|x|$, we say M *accepts* x *uniquely*.

Definition. $A(x)$, the *automatic complexity* of x, is the smallest number of states in any DFA M that accepts x uniquely.

- There may be many such DFA's with the smallest number of states.
- We do not care how M behaves on strings that are shorter or longer than x.
- We can view the map $x \to (M, |x|)$ as a compression algorithm for x.

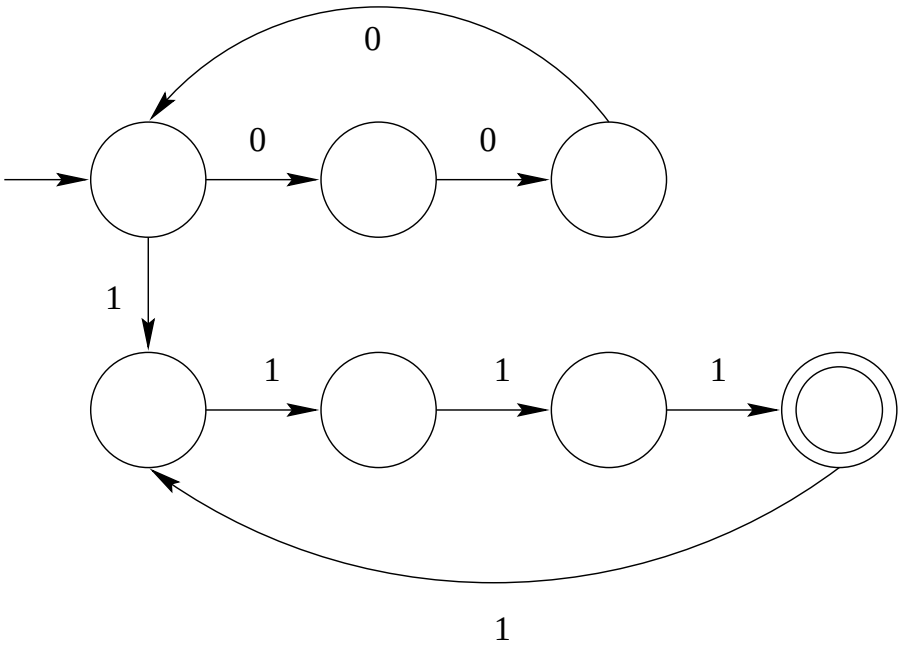
Basic Properties of Automatic Complexity

- $A(x) \leq |x| + 2$, since we can accept a string with a chain of $|x| + 1$ states, plus one more for a dead state.
- Hence A is computable, since we can simply
  - Enumerate all DFA's with $\leq |x| + 2$ states, and
  - For each such automaton, we check if it accepts x and if it rejects all other strings of length $|x|$.
- Better algorithms are possible — for example, it is possible, given a DFA and a string x, to determine efficiently if x is accepted uniquely.
- But we still do not know an efficient algorithm for computing A, or have a proof that computing A is, say, NP-hard
- On the other hand, given a description of a DFA M which uniquely accepts x, and the length $n = |x|$, we can efficiently recover x.

How to Save States

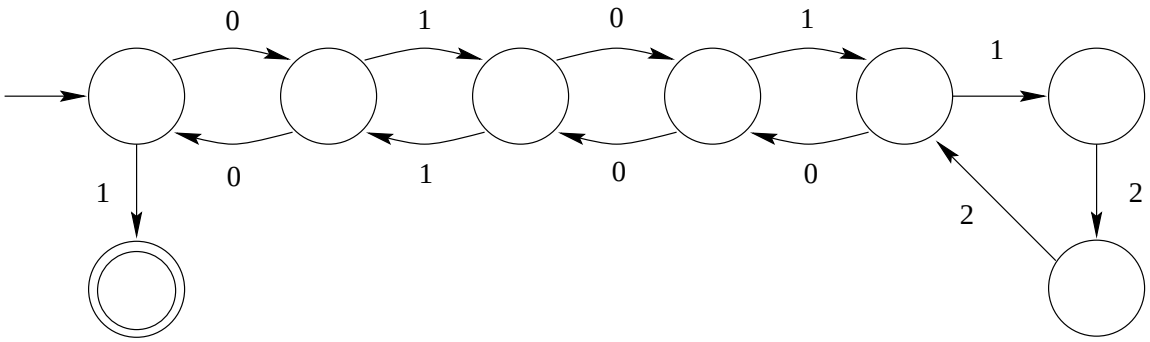
You can save states with loops:

For example, the automaton below shows that $A(0^9 1^8) \leq 8$. (Unspecified transitions go to a "dead state" which is not shown.)



How to Save States

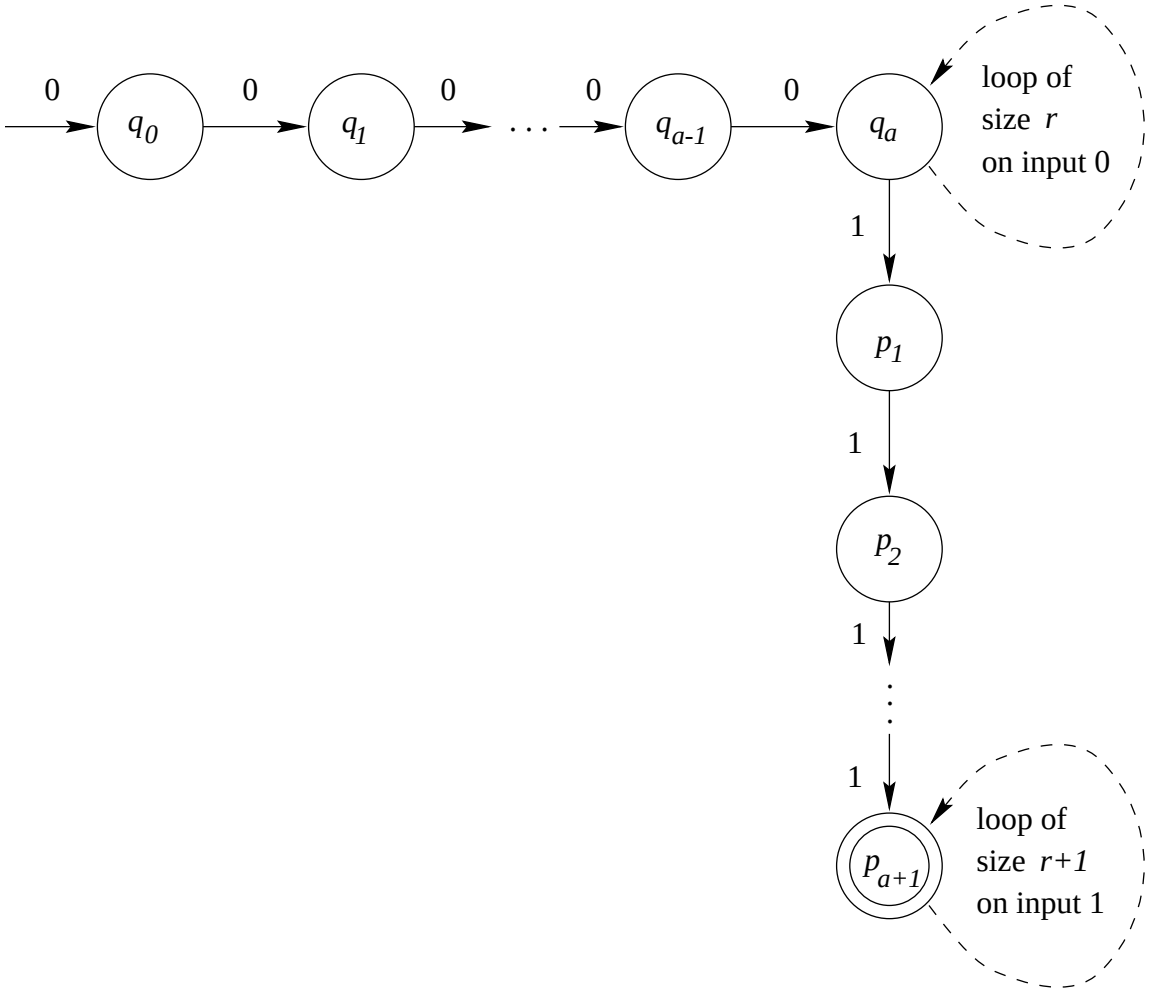
You can reuse states, if the string is of the form $x\, y\, z\, \overline{y}^R\, w$, as follows:



Some Specific Examples

Theorem. We have $A(0^n 1^n) = O(\sqrt{n})$.

Proof. Assume $n \geq 1$. Let $r = \lfloor \sqrt{n} \rfloor$, so $r^2 \leq n < (r+1)^2$. Write $n = r^2 + a$. Then $0 \leq a \leq 2r$ and $r \geq 1$. Then we can accept $0^n 1^n$ with an automaton of the form below. (Unspecified transitions go to a "dead state" which is not shown.)



This automaton does indeed accept 0^n1^n because

1. We go from state q_0 to state q_a on 0^a;
2. We then go around the loop at q_a r times;
3. Next on 1^a we go from q_a to p_{a+1};
4. Finally, we go around the loop at p_{a+1} $r-1$ times.

This path accepts $0^a(0^r)^r11^a(1^{r+1})^{r-1} = 0^{r^2+a}1^{r^2+a}$.

On the other hand, we claim that this DFA accepts no other string of length $2n$. Suppose it did. Then any accepting path must go around the loop on q_a b times and the loop on p_{a+1} c times. Then

$$2n = a + br + a + 1 + c(r+1).$$

Since $n = r^2 + a$, it follows that $2r^2 - 1 = br + c(r+1)$. Reducing modulo r, we get $c \equiv -1 \pmod{r}$. Thus $c \in \{r-1, 2r-1, \ldots\}$. But if $c \geq 2r-1$ then the string would be of length $\geq (2r-1)(r+1) + 2a + 1 = 2r^2 + r - 1 + 2a + 1 \geq 2n + r > 2n$, a contradiction.

Finally, our automaton uses $a + 1 + r - 1 + a + 1 + r = 2r + 2a + 1 \leq 6r + 1 \leq 6\sqrt{n} + 1$ states. ∎

Lower Bound for $A(0^n1^n)$

We now show that the bound of $O(\sqrt{n})$ is tight. We use the following lemma.

Lemma. Let c, d be integers ≥ 1. Suppose the linear diophantine equation $N = xc + yd$ is solvable in integers, i.e., suppose $\gcd(c, d) \mid N$. If $N > 2cd - c - d$, then the linear diophantine equation $N = xc + yd$ has at least two solutions in non-negative integers x, y.

Theorem. Any DFA that uniquely accepts 0^n1^n must have at least $\sqrt{n} - 1$ states.

Generalization to $A(a_1^n a_2^n \cdots a_k^n)$

Can we generalize our result for $0^n 1^n$ to strings of the form $a_1^n a_2^n \cdots a_k^n$?

To do so, we would need loops on each of the symbols $a_1, a_2, \ldots, a_k$. If the sizes of the loop corresponding to a_i is s_i, and we go around it e_i times, then we want $e_i s_i$ to be less than but close to n, say $n - l_i$, and we want to have exactly one solution to the equation

$$\sum_{1 \le i \le k} e_i s_i = kn - \sum_{1 \le i \le k} l_i.$$

One way to do this is to pick k numbers, relatively prime in pairs, say n_1, n_2, ..., n_k, each less than $n^{1/k}$ but relatively close to it in size. Let $N = n_1 n_2 \cdots n_k$. Let the loop sizes be N/n_i for $1 \le i \le k$. It is now possible to show that the linear diophantine equation

$$e_1(N/n_1) + e_2(N/n_2) + \cdots + e_k(N/n_k) = C$$

has a unique solution

$$(e_1, e_2, \ldots, e_k) = (n_1 - 1, n_2 - 1, \ldots, n_k - 1)$$

for suitably chosen C.

Modulo the existence of the relatively prime numbers, we have shown

Generalization to $A(a_1^n a_2^n \cdots a_k^n)$

Theorem. Let $a_1, a_2, \ldots, a_k$ be k distinct symbols. Then $A(a_1^n a_2^n \cdots a_k^n) = O(n^{1-1/k})$.

But it still remains to show that one can find the k relatively prime numbers.

It suffices to show that for each integer $k \geq 1$ there exists a number $f(k)$ such that every $f(k)$ consecutive positive integers contains a subset of size k that is pairwise relatively prime. This can be done as follows:

Lemma. Let p_n denote the n'th prime, with $p_1 = 2$. Define $Q_n = p_1 p_2 \ldots p_n$ for $n \geq 1$. Then we may take $f(k) = Q_k$.

Proof. We claim that for each integer $r \geq 0$ and $n \geq 1$ the sets $U_r := \{rQ_n + 2, rQ_n + 3, \ldots, rQ_n + p_n\}$ and $V_r := \{(r+1)Q_n - 2, (r+1)Q_n - 3, \ldots, (r+1)Q_n - p_n\}$ are individually pairwise relatively prime. U_r. ∎

Jacobsthal's Function

We have seen that we have $f(k) \leq p_1 p_2 \cdots p_k$, the product of the first k primes. By the prime number theorem we know that $C = e^{k(\log k)(1+o(1))}$. This exponential bound can be significantly improved.

Let $n \geq 1$ be an integer, and let

$$c_1 = 1 < c_2 < \cdots < c_r$$

be the integers in the range $[1, n]$ that are relatively prime to n, where $r = \varphi(n)$.

Then Jacobsthal's function $g(n)$ is defined to be

$$\max_{2 \leq i \leq r} (c_i - c_{i-1}),$$

i.e., the size of the maximum gap between two consecutive numbers relatively prime to n.

Also define

$$C(r) = \max_{\omega(n)=r} g(n),$$

where ω counts the number of distinct prime divisors.

Jacobsthal's function

Let I be a set of consecutive positive integers. We call a subset of I that is pairwise relatively prime a $clique$.

Define $f(m)$ to be the least integer n such that every set of n consecutive positive integers contains a clique of size m.

Theorem. We have $f(m) = C(m-1)$ for $m \geq 1$.

Proof.

- Suppose $C(m-1) = s$.
- Then there exist $m-1$ prime numbers $p_1, p_2, \ldots, p_{m-1}$ such that $g(p_1 p_2 \cdots p_{m-1}) = s$.
- Then there exists an integer t such that the numbers $t+1, t+2, \ldots, t+s-1$ all have a factor in common with $p_1 p_2 \cdots p_{m-1}$, i.e., each integer in

  $$\{t+1, t+2, \ldots, t+s-1\}$$

  is divisible by at least one of the primes $p_1, p_2, \ldots, p_{m-1}$.
- Hence the largest clique in $\{t+1, t+2, \ldots, t+s-1\}$ is of size $\leq m-1$.
- Therefore $f(m) > s-1$, and so $f(m) \geq C(m-1)$.

Jacobsthal's function

- Now suppose $f(m) = s$.
- Then there exists a set $s-1$ consecutive positive integers, say $S = \{t+1, t+2, \ldots, t+s-1\}$, which contains no clique of size m.
- However

  $$S \cup \{t+s\} = \{t+1, t+2, \ldots, t+s-1, t+s\}$$

  must contain a clique of size m, so S must contain a clique of size $m-1$, say $T = \{a_1, a_2, \ldots, a_{m-1}\}$.
- Without loss of generality we may assume $1 \notin T$.
  - For if $1 \in T$, then $1 \in S$ and hence $t = 0$.
  - Then $\{1, 2, \ldots, s\}$ contains a clique of size m.
  - Now $\{2, \ldots, s\}$ cannot contain a clique of size $\geq m$, for if it did, $\{1, 2, \ldots, s\}$ would contain a clique of size $m+1$, and so $\{1, 2, \ldots, s-1\}$ would contain a clique of size m, a contradiction.
  - Thus in this case we may instead take $S = \{2, 3, \ldots, s\}$.
- Since $1 \notin T$, at least one prime divides each member of T.
- Further, we cannot have the same prime dividing two different members of T, for then T would not be a clique.

- Thus there is a set of $m-1$ distinct primes, say $P = \{p_1, p_2, \ldots, p_{m-1}\}$, such that each prime of P divides exactly one element of T.
- Now suppose there exists $a \in S$ such that no prime of P divides a.
- Then $T \cup \{a\}$ would be a clique of size m, a contradiction.
- It follows that $\gcd(a, p_1 p_2 \cdots p_{m-1}) > 1$ for all $a \in S$.
- But then we have $g(p_1 p_2 \cdots p_{m-1}) \geq s$.
- Now $C(m-1) \geq g(p_1 p_2 \cdots p_{m-1})$, so $C(m-1) \geq f(m)$.

Thus we have proved $f(m) = C(m-1)$. ∎

We now use a result of IWANIEC, which shows that $C(m) = O(m^2(\log m)^2)$. It follows that $f(m) = O(m^2(\log m)^2)$.

Infinite Words

By an infinite word we mean a one-sided, right-infinite word, i.e., a map from $\mathbb{N}$ to Σ.

For an infinite word $\mathbf{x}$ we we are interested in computing

$$I(\mathbf{x}) = \liminf_{x \text{ is a prefix of } \mathbf{x}} \frac{A(x)}{|x|}$$

and

$$S(\mathbf{x}) = \limsup_{x \text{ is a prefix of } \mathbf{x}} \frac{A(x)}{|x|}.$$

for "interesting" infinite words $\mathbf{x}$.

Theorem. Let $\mathbf{t} = 0110100110010110\cdots$ be the infinite Thue-Morse word. Then

$$I(\mathbf{t}) \geq \frac{1}{3}$$

and

$$S(\mathbf{t}) \leq \frac{2}{3}.$$

Proof. The lower bound for $I(\mathbf{t})$ follows immediately from a previous result, since the Thue-Morse word is cube-free.

For the upper bound, we break the argument up as follows. Let t be a prefix of $\mathbf{t}$ of length m. We claim that we can accept t using $h(m)$ states, where h is as follows:

m	$h(m)$
$2 \cdot 2^{2n} \leq m \leq 3 \cdot 2^{2n}$	$m + 3 - 2 \cdot 2^{2n}$
$3 \cdot 2^{2n} < m < 4 \cdot 2^{2n}$	$2 \cdot 2^{2n} + 2$
$4 \cdot 2^{2n} \leq m < 5 \cdot 2^{2n}$	$m + 2 - 2 \cdot 2^{2n}$
$5 \cdot 2^{2n} \leq m \leq 6 \cdot 2^{2n}$	$m + 1 - 2 \cdot 2^{2n}$
$6 \cdot 2^{2n} < m < 8 \cdot 2^{2n}$	$4 \cdot 2^{2n} + 2$

The formal proof is somewhat tedious, so we simply illustrate the construction with representative values of m. As usual, unspecified transitions go to a "dead state" which is not shown.

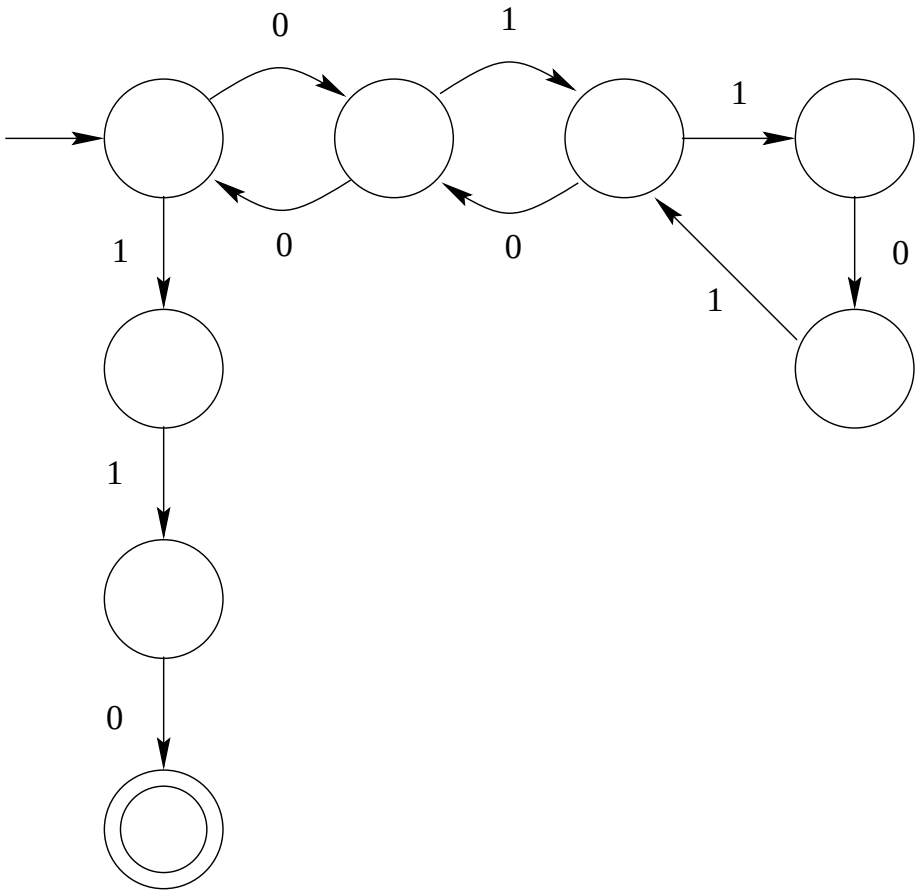


Figure 1: Automaton uniquely accepting t_{10}

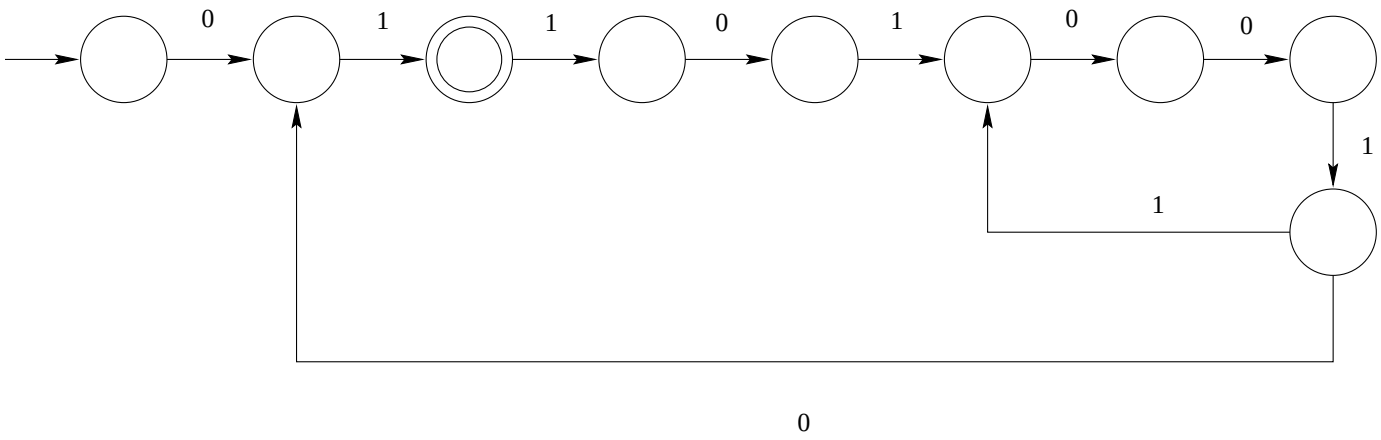


Figure 2: Automaton uniquely accepting t_{14}