The Main Themes

- periodicity
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- patterns and pattern avoidance
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- periodicity
- patterns and pattern avoidance
- equations in words
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- periodicity
- patterns and pattern avoidance
- equations in words
- infinite words and their properties
Some notation

\(\Sigma\) - a finite nonempty set of symbols - the \textit{alphabet}
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word - a finite or infinite list of symbols chosen from $\Sigma$
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\( \Sigma^\infty = \Sigma^\ast \cup \Sigma^\omega \)
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$w = a_1 a_2 \cdots a_n$
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prefix, suffix, factor, subword
Algebraic framework

- semigroup: concatenation is multiplication, associative
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- monoid: semigroup + identity element ($\epsilon$)
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- monoid: semigroup + identity element \((\epsilon)\)
- free monoid: no relations among elements
- group: add inverses of elements \(a^{-1}\)
Periodicity

words - fundamentally noncommutative
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words - fundamentally noncommutative

casebook ≠ bookcase
Periodicity

words - fundamentally noncommutative

casebook $\neq$ bookcase

When do words commute?
words - fundamentally noncommutative

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Here are two words that “almost” commute:
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When do words commute?
Here are two words that “almost” commute:

$w = 01010$ and $x = 01011010$

$wx = 0101001011010$
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When do words commute?

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\[ w = 01010 \text{ and } x = 01011010 \]

\[ wx = 0101001011010 \]
\[ xw = 0101101001010 \]

By the way, this raises the question: can the Hamming distance between \( wx \) and \( xw \) be 1? It can’t; there is a one-line proof.
What are the solutions to $x^2 = y^3$ in words?

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First Theorem of Lyndon-Schützenberger

**Theorem**

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Let \( x, y \in \Sigma^+ \). Then the following three conditions are equivalent:

1. \( xy = yx \);
2. There exist \( z \in \Sigma^+ \) and integers \( k, l > 0 \) such that \( x = z^k \) and \( y = z^l \);
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Let $x, y \in \Sigma^+$. Then the following three conditions are equivalent:

1. $xy = yx$;
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3. There exist integers $i, j > 0$ such that $x^i = y^j$. 
Second Theorem of Lyndon-Schützenberger

Under what conditions can a string have a nontrivial proper prefix and suffix that are identical?
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Examples in English: reader — begins and ends with r
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Examples in English: reader — begins and ends with r
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The answer is given by the following theorem.

**Theorem**

Let $x, y, z \in \Sigma^+$. Then $xy = yz$ if and only if there exist $u \in \Sigma^+$, $v \in \Sigma^*$, and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u = u(vu)^e$. 
Primitive words

We say a word \( x \) is a \textit{power} if it can be expressed as \( x = y^n \) for some \( y \neq \epsilon, \ n \geq 2 \).
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Open question: is the set of primitive binary words a CFL?
Conjugates

A word \( w \) is a conjugate of a word \( x \) if \( w \) can be obtained from \( x \) by cyclically shifting the letters.
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A conjugate of a \( k \)-th power is a \( k \)-th power of a conjugate.
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**Theorem:** Every finite word has a unique factorization as the product of Lyndon words \( w_1w_2\cdots w_n \), where \( w_1 \geq w_2 \geq w_3 \cdots w_n \).
We say a word $w$ is a $p/q$ power, for integers $p \geq q \geq 1$, if

$$w = x^{\lfloor p/q \rfloor}x'$$

for a prefix $x'$ of $x$ such that $|w|/|x| = p/q$. 
Fractional powers

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If \( w = x^{\lfloor p/q \rfloor} x' \) is a \( p/q \) power, then we call \( x \) a *period* of \( w \).

Often the word *period* is used to refer to \( |x| \).
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Often the word *period* is used to refer to $|x|$.

If a word $w$ is a $p/q > 1$ power, then it begins and ends with some nonempty string. Such a string is also called *bordered*. Otherwise it is *unbordered*. 
Unbordered words

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However, there are asymptotically $c_k k^n$ such words, where $c_k$ is a constant that tends to 1 as $k$ tends to $\infty$. 
Duval’s conjecture

Let $\mu(w)$ be the length of the longest unbordered factor of $w$. 
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Duval’s conjecture

Let $\mu(w)$ be the length of the longest unbordered factor of $w$. Let $p(w)$ be the length of the longest period of $w$. Duval conjectured that if $|w| \geq 3\mu(w)$, then $\mu(w) = p(w)$. This was proved by Harju & Nowotka, and S. Holub. The result has been improved to $|w| \geq 3\mu(w) - 2 \implies \mu(w) = p(w)$. 
The Fine-Wilf theorem

Theorem

Let $w$ and $x$ be nonempty words. Let $y \in w\{w, x\}^\omega$ and $z \in x\{w, x\}^\omega$. Then the following conditions are equivalent:

(a) $y$ and $z$ agree on a prefix of length at least $|w| + |x| - \gcd(|w|, |x|)$;
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(c) $y = z$.

(c) $\implies$ (a): Trivial.
We’ll prove (a) $\implies$ (b) and (b) $\implies$ (c).
Proof.
(a) $y$ and $z$ agree on a prefix of length at least
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Then we prove that \( y \) and \( z \) differ at a position
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The proof is by induction on $|w| + |x|$.
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The base case is $|w| + |x| = 2$. Then $|w| = |x| = 1$, and $|w| + |x| - \gcd(|w|, |x|) = 1$. Since $wx \neq xw$, we must have $w = a$, $x = b$ with $a \neq b$. Then $y$ and $z$ differ at the first position.
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We prove it for $|w| + |x| = k$.

If $|w| = |x|$ then $y$ and $z$ must disagree at the $|w|$'th position or earlier, for otherwise $w = x$ and $wx = xw$; since $|w| \leq |w| + |x| - \gcd(|w|, |x|) = |w|$, the result follows.
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So, without loss of generality, assume $|w| < |x|$.

If $w$ is not a prefix of $x$, then $y$ and $z$ disagree on the $|w|$’th position or earlier, and again $|w| \leq |w| + |x| - \gcd(|w|, |x|)$. 

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Write $x = wt$ for some nonempty word $t$.

Now any common divisor of $|w|$ and $|x|$ must also divide $|x| - |w| = |t|$, and similarly any common divisor of both $|w|$ and $|t|$ must also divide $|w| + |t| = |x|$. So $\gcd(|w|, |x|) = \gcd(|w|, |t|)$. 
Now $wt \neq tw$, for otherwise we have $wx = wwt = wtw = xw$, a contradiction.
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Then $y = wwy \cdots$ and $z = wty \cdots$. By induction (since $|w| + |t| < k$) $w^{-1}y$ and $w^{-1}z$ disagree at position $|w| + |t| - \gcd(|w|, |t|)$ or earlier.
Now \( wt \neq tw \), for otherwise we have \( wx = wwt = wtw = xw \), a contradiction.

Then \( y = ww \cdots \) and \( z = wt \cdots \). By induction (since \( |w| + |t| < k \) \( w^{-1}y \) and \( w^{-1}z \) disagree at position \( |w| + |t| - \gcd(|w|, |t|) \) or earlier.

Hence \( y \) and \( z \) disagree at position
\[
2|w| + |t| - \gcd(|w|, |t|) = |w| + |x| - \gcd(|w|, |x|) \] or earlier.
(b) $\implies$ (c): If $wx = xw$, then by the theorem of Lyndon-Schützenberger, both $w$ and $x$ are in $u^+$ for some word $u$. Hence $y = u^\omega = z$. ■
The proof also implies a way to get words that optimally “almost commute”, in the sense that $xw$ and $wx$ should agree on as long a segment as possible.
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**Theorem**

For each \( m, n \geq 1 \) there exist words \( x, w \) of length \( m, n \), respectively, such that \( xw \) and \( wx \) agree on a prefix of length \( m + n - \gcd(m, n) - 1 \) but differ at position \( m + n - \gcd(m, n) \).
Finite Sturmian words

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Many authors have worked on generalizations to multiple periods: Castelli, Mignosi, & Restivo, Simpson & Tijdeman, Constantinescu & Ilie, ...
Patterns and pattern avoidance

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But over a 3-letter alphabet, it is possible to create arbitrarily long words (or — what is equivalent — an infinite word) with no square factors at all. Such a word is called \textit{squarefree}.
The story begins with Axel Thue in 1906.

He noticed that over a 2-letter alphabet, every word of length $\geq 4$ contains a square: either $0^2$, $1^2$, $(01)^2$ or $(10)^2$.

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Patterns and pattern avoidance

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It is based on the Thue-Morse sequence.
The Thue-Morse morphism

Morphism: a map $h$ from $\Sigma^*$ to $\Delta^*$ such that

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We write $h^i = h(h(h(\cdots))).$
Morphic words

If a nonerasing morphism has the property that \( h(a) = ax \), then iterating \( h \) produces an infinite word

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h^\omega(a) = ax h(x) h^2(x) h^3(x) \cdots.
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Also rediscovered by Marston Morse, Max Euwe, Solomon Arshon, and the Danish composer Per Nørgård.
Properties of the Thue-Morse word

An overlap is a word of the form $axaxa$, where $a$ is a single letter and $x$ is a word.
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Theorem

*The Thue-Morse word $t$ is overlap-free.*
From overlap-free to squarefree

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```
0 11 0 1 0 0 11 0 0 1 0 11 0 1 0 0 1 0 11 ...
```

This is squarefree, as a square in this word implies and overlap in the Thue-Morse word.
Enumeration of power-free words

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Infinite - countable or uncountable
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For overlap-free words over $\{0, 1\}$ there is a factorization theorem of Restivo and Salemi that implies only polynomially-many of length $n$.
For squarefree words over $\{0, 1, 2\}$ there are exponentially many.
Dejean’s Conjecture

Given an alphabet $\Sigma$ of cardinality $k$, we can try to find the optimal (fractional) exponent $\alpha_k$ avoidable by infinite words over $\Sigma$. 
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Still open: many other variants of Dejean where the length of the period is also taken into account.
More general patterns

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Not all patterns are avoidable — even if the alphabet is arbitrarily large.

For example - it is impossible to avoid $xyx$, since every sufficiently long string $z$ will contain three occurrences of some letter $a$, say $z = rasatau$, and then we can let $x = a$, $y = sat$, both $x$ and $y$ are nonempty.
Avoiding general patterns

Given a pattern, it is decidable (via Zimin’s algorithm) if it is avoidable over some alphabet.
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However, we do not have a general procedure to decide if a given pattern is avoidable over a fixed alphabet.
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For example, *interessierten* is an abelian square in German, as *sierten* is a permutation of *interes*.

In a similar way, we can define abelian cubes as words of the form $xx'x''$ where both $x'$ and $x''$ are permutations of $x$. 
Abelian powers: summary of results

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An open problem: is it possible to avoid, over a finite subset of \( \mathbb{N} \), patterns of the form \( xx' \) where \( |x| = |x'| \) and \( \sum x = \sum x' \)?
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Also still open: fractional version of abelian powers
A generalization of abelian powers

involution: $h^2(x) = x$ for all words $x$
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Can avoid some patterns involving involution, but not others
Equations in words

Example 1:

\[ abX = Xba. \]
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The solutions are $X = a^i$, $Y = (a^i b)^j a^i$ for $i \geq 0, j \geq 0.$
Example 3: Fermat’s equation for words: $x^i y^j = z^k$
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The only solutions for $i, j, k \geq 2$ are when $x, y, z$ are all powers of a third word.

Example 4:

$XYZ = ZVX$: many solutions, but not expressible by formula with integer parameters.
More generally, given an equation in words and constants, there is an algorithm (Makanin’s algorithm) that is guaranteed to find a solution if one exists.
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automatic: image, under a coding, of a fixed point of a uniform morphism
Other important infinite words

Sturmian words: exactly $n + 1$ factors of length $n$
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Episturmian words: natural generalization of Sturmian words to larger alphabets
More infinite words

Toeplitz words: generated by starting with a periodic word with “holes”; then inserting another periodic word with holes into that, etc.
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- The sequence 1221121221\cdots that encodes its own sequence of run lengths
Properties of infinite words

recurrence - every factor that occurs, occurs infinitely often
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uniform recurrence - recurrent, plus distance between two consecutive occurrences of the same factor of length $n$ is bounded, for all $n$
“subword” complexity - given an infinite word $w$, count the number of distinct factors of length $n$ in $w$
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Subword complexity

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- \( O(n) \) for automatic sequences
- \( n + 1 \) for Sturmian words
- \( O(n^2) \) for morphic words
- A classification of possible growth rates exists
A deterministic finite automaton with output (DFAO) is a 6-tuple: \((Q, \Sigma, \delta, q_0, \Delta, \tau)\), where \(\Delta\) is the finite output alphabet and \(\tau : Q \rightarrow \Delta\) is the output mapping.
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Next, we decide on an integer base \(k \geq 2\) and represent \(n\) as a string of symbols over the alphabet \(\Sigma = \{0, 1, 2, \ldots, k - 1\}\).
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To compute $f_n$, given an automaton $M$, express $n$ in base-$k$, say,

$$a_r a_{r-1} \cdots a_1 a_0,$$

and compute

$$f_n = \tau(\delta(q_0, a_r a_{r-1} \cdots a_1 a_0)).$$
Automatic sequences

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- Any sequence that can be computed in this way is said to be \(k\)-automatic.
The Thue-Morse automaton

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Robustness

the order in which the base-\(k\) digits are fed into the automaton in does not matter (provided it is fixed for all \(n\));
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- other representations also work (such as expansion in base-$(−k)$);
- automatic sequences are closed under many operations, such as shift, periodic deletion, $q$-block compression, and $q$-block substitution.
- if a symbol in an automatic sequence occurs with well-defined frequency $r$, then $r$ is rational.
Christol’s theorem

Theorem (Christol [1980]). Let \((u_n)_{n \geq 0}\) be a sequence over

\[
\Sigma = \{0, 1, \ldots, p - 1\},
\]

where \(p\) is a prime. Then the formal power series

\[
U(X) = \sum_{n \geq 0} u_n X^n
\]

is algebraic over \(GF(p)[X]\) if and only if \((u_n)_{n \geq 0}\) is \(p\)-automatic.
Christol’s theorem: example

Let \((t_n)_{n \geq 0}\) denote the \textsc{Thue-Morse} sequence.
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Let \((t_n)_{n \geq 0}\) denote the THUE-MORSE sequence. Then \(t_n = \text{sum of the bits in the binary expansion of } n, \text{ mod } 2\).
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\[
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Hence \((1 + X)^3 A^2 + (1 + X)^2 A + X = 0\).
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- Is there a word over a finite subset of \( \mathbb{N} \) that avoids \( xx' \) with \( |x| = |x'| \) and \( \sum x = \sum x' \)?
For Further Reading