Remarks on Inferring Integer Sequences

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Introduction

What are the rules behind the following integer sequences?

- $1, 2, 3, 4, 5, 6, 7, 8, \ldots$
- $2, 5, 10, 17, 26, 37, 50, 65, \ldots$
- $2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, \ldots$
- $0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$
- $0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 4, 5, \ldots$
- $0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30, \ldots$

Given an integer sequence, how can we determine what it is?

The question is ill-posed: can only look at a finite number of terms, and any such sequence has an infinite number of potential extensions.

Two Approaches to Sequence Recognition

- One can mathematically define a large class of sequences and then try to determine membership in that class
  - Dana Angluin (UC Berkeley Technical Report, 1974)
  - A. K. Dewdney (*Scientific American*, Mathematical Recreations, March 1986)
  - Bhansali and Skiena (*Computational Support for Discrete Mathematics*, 1994)
  - Sloane and Plouffe's `SuperSeeker` program (`superseeker@research.att.com`)

- One can collect sequences from the literature and then try to express the target sequence in terms of known sequences
  - Peter Liu (Master's Essay, University of Waterloo, 1994)
  - Sloane and Plouffe's `SuperSeeker` program

Two Neglected Classes of Sequences

- The k-automatic sequences
  - form about 3% of the sequences in the Sloane-Plouffe table
- The k-regular sequences
  - form about 7% of the sequences in the Sloane-Plouffe table

Basics of Finite Automata

- If Σ is a finite set of symbols, then by Σ^* we mean the free monoid over Σ (set of all finite strings of symbols chosen from Σ);
- A *language* is a subset of Σ^*.
- a *finite automaton* is a simple model of a computer
- formally it is a quintuple: $M = (Q, \Sigma, \delta, q_0, F)$ where:
  - Q is a finite set of *states*;
  - Σ is a finite set of symbols, called the *input alphabet*;
  - $q_0 \in Q$ is the *start state*;
  - $F \subseteq Q$ is the set of *final states*;
  - $\delta : Q \times \Sigma \rightarrow Q$ is the *transition function*
- The *language accepted by M* is denoted by $L(M)$ and is given by $\{w \in \Sigma^* \mid \delta(q_0, w) \in F\}$.

Example of a Finite Automaton

<u>Automata as Computers of Sequences</u>

- First, we can generalize our notion of automaton to provide an output, not simply accept/reject.
- Formally, we define a *deterministic finite automaton with output* (DFAO) as a sextuple: $(Q, \Sigma, \delta, q_0, \Delta, \tau)$, where Δ is the finite *output alphabet* and $\tau : Q \to \Delta$ is the *output mapping*.
- Next, we decide on a integer base $k \geq 2$ and represent n as a string of symbols over the alphabet $\Sigma = \{0, 1, 2, \ldots, k-1\}$.
- To compute f_n, given an automaton M, express n in base-k, say, $a_r a_{r-1} \cdots a_1 a_0$, and compute $f_n = \tau(\delta(q_0, a_0 a_1 \cdots a_{r-1} a_r))$.
- Any sequence that can be computed in this way is said to be <u>k-automatic</u>.

k-Automatic Sequences

A sequence $(a_n)_{n \geq 0}$ is said to be k-automatic if, a_n is a finite-state ("automatic") function of the base-k representation of n.

Example. The following automaton generates the Rudin-Shapiro sequence:

To compute r_n, expand n in base-2, and then input the bits of n into the automaton, starting with the least significant bit, transiting from state to state. When last state is encountered, output is specified in the state.

The Thue-Morse Sequence

- Introduced by Axel Thue (1863–1922).
- $t_n =$ sum of bits of n (base 2), taken modulo 2.
- First few terms: 0 1 1 0 1 0 0 1 1 0 0 1 0 $\cdots$

Example 1. An unusual infinite product. Define $a_n = (-1)^{t_n}$ for $n \geq 0$. Then

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{a_n} = \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{7}{8} \cdots = \frac{\sqrt{2}}{2}.$$

Example 2. A converse of sorts to Example 1. Define $b_0 = 1$, and

$$b_n = \begin{cases} 1, & \text{if } \prod_{0 \leq i < n} \left(\frac{2i+1}{2i+2}\right)^{b_i} > \sqrt{2}/2; \\ -1, & \text{if } \prod_{0 \leq i < n} \left(\frac{2i+1}{2i+2}\right)^{b_i} < \sqrt{2}/2. \end{cases}$$

Then $a_n = b_n$.

The Thue-Morse sequence (u_n) continued

Example 3. Prouhet's result of 1851 on "multigrades". Separate the integers in the set

$$S_n = \{0, 1, 2, \ldots, 2^n - 1\}$$

into two subsets:

$$T_n = \{i \in S_n \ : \ t_i = 0\}$$

and

$$U_n = \{i \in S_n \ : \ t_i = 1\}.$$

Then

$$\sum_{k \in T_n} k^j = \sum_{\ell \in U_n} \ell^j$$

for $j = 0, 1, \ldots, n - 1$.

Example:

$$0^i + 3^i + 5^i + 6^i + 9^i + 10^i + 12^i + 15^i =$$

$$1^i + 2^i + 4^i + 7^i + 8^i + 11^i + 13^i + 14^i$$

for $i = 0, 1, 2, 3$.

The Rudin-Shapiro Sequence (u_n)

- Define $u_n = (-1)^{r_n}$, where r_n counts the number of (possibly overlapping) occurrences of the block '11' in the binary representation of n.
- This sequence was introduced by Rudin and Shapiro, independently.

Example 1. It is easy to prove that, for any sequence $(a_n)_{n\geq 0}$ of $+1$'s and -1's, we have

$$\sup_\theta \left| \sum_{0\leq k<n} a_k e^{ik\theta} \right| \geq \sqrt{n}.$$

On the other hand, it can be shown that for "almost all" sequences $(a_n)_{n\geq 0}$ we have

$$\sup_\theta \left| \sum_{0\leq k<n} a_k e^{ik\theta} \right| = O(\sqrt{n \log n}).$$

Rudin and Shapiro (independently) proved in the 1950's that

$$\sup_\theta \left| \sum_{0\leq k<n} u_k e^{ik\theta} \right| = O(\sqrt{n}).$$

The Rudin-Shapiro Sequence, Continued

Example 2. Consider a path visiting lattice points in the plane. Start at the origin and make a first move to $(0, 1)$. At step n, turn "left" or "right" 90° according to the following rule:

- "left", if $r(n) - r(n-1) + n \equiv 0 \pmod 2$;
- "right", if $r(n) - r(n-1) + n \equiv 1 \pmod 2$.

We get a spacefilling curve that visits every lattice point in 1/8 of the plane exactly once.

Robustness of the Notion of Automatic Sequence

- the order in which the base-k digits are fed into the automaton in does not matter (provided it is fixed for all n);
- other representations also work (such as expansion in base-$(-k)$);
- automatic sequences are closed under many operations, such as shift, periodic deletion, q-block compression, and q-block substitution.
- automatic sequences are also closed under uniform transduction.
  - a uniform finite-state transducer is like an automaton, but outputs s symbols at each transition

Properties of Automatic Sequences

Definition.
The *k-kernel* of a sequence $(a_n)_{n\geq 0}$ is the set of subsequences

$$\{(a_{k^r n+c})_{n\geq 0} \ : \ r \geq 0, \ 0 \leq c < k^r\}.$$

Cobham's 1st Theorem. A sequence is k-automatic if and only if its k-kernel is finite.

Definition.
A *homomorphism* $\varphi : \Sigma^* \to \Sigma^*$ is a map satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Sigma^*$. If $|\varphi(a)| = k$ for all $a \in \Sigma$, then we say φ is *k-uniform*. A *coding* is a 1-uniform homomorphism.

Cobham's 2nd Theorem. A sequence is k-automatic if and only if it is the image (under a coding) of a fixed point of a k-uniform homomorphism.

Example. The Thue-Morse sequence is the fixed point of the map $0 \to 01$, $1 \to 10$ that starts with 0.

The Theorem of Christol-Kamae-Mendès France-Rauzy

Theorem. (Christol, Kamae, Mendès France, Rauzy, 1980). Let $(u_n)_{n\geq 0}$ be a sequence over

$$\Sigma = \{0, 1, \ldots, p-1\},$$

where p is a prime. Then the formal power series $U(X) = \sum_{n\geq 0} u_n X^n$ is algebraic over $GF(p)[X]$ if and only if $(u_n)_{n\geq 0}$ is p-automatic.

Example.

Let, as before, $(t_n)_{n\geq 0}$ denote the Thue-Morse sequence, i.e., t_n = sum of the bits in the binary expansion of n, mod 2. Then $t_{2n} \equiv t_n$ and $t_{2n+1} \equiv t_n + 1$. If we set $A(X) = \sum_{n\geq 0} t_n X^n$, then

$$\begin{aligned} A(X) &= \sum_{n\geq 0} t_{2n} X^{2n} + \sum_{n\geq 0} t_{2n+1} X^{2n+1} \\ &= \sum_{n\geq 0} t_n X^{2n} + X \sum_{n\geq 0} t_n X^{2n} + X \sum_{n\geq 0} X^{2n} \\ &= A(X^2) + XA(X^2) + X/(1-X^2) \\ &= A(X)^2(1+X) + X/(1+X)^2. \end{aligned}$$

Hence $(1+X)^3 A^2 + (1+X)^2 A + X = 0$.

Inferring Automatic Sequences

- Can one infer a k-automatic sequence, given the first few terms?
- If a sequence *is* k-automatic, and is generated by an automaton with $\leq r$ states, then given the first k^{2r-2} terms, one can correctly and efficiently predict all future terms of the sequence.
- In practice k and r are usually small, and the correct automaton can often be guessed with far fewer terms.
- The automaton can be inferred purely mechanically, by examining the k-kernel, and declaring two members to be equal if they agree on the terms actually known.
- If a sequence is *not* k-automatic, then it is possible to have two genuinely different elements of the k-kernel agree on thousands or millions of terms before a distinguishing element is found.
- However, this rarely occurs in practice.

An Amazing Non-Automatic Sequence

Take the Thue-Morse sequence

$$(t_n)_{n\geq 0} = 011010011001\cdots,$$

and create a new sequence

$$(u_n)_{n\geq 0} = 12112221121\cdots$$

that counts the lengths of blocks of identical symbols in $(t_n)_{n\geq 0}$.

Then it can be shown that (u_n) is not a 2-automatic sequence, (but the proof is not easy at all).

An Amazing Non-Automatic Sequence

However, the sequence (u_n) comes very "close" to being 2-automatic, in that to distinguish two sequences in the kernel, one must look at very large values of n. For example, $u_{8n} = u_{32n}$ for $0 \le n \le 14562$, but not for $n = 14563$. Similarly, $u_{16n+1} = u_{64n+1}$ for $0 \le n \le 1864134$, but not for $n = 1864135$.

A complete understanding of the behaviour of this sequence is still not at hand, but it depends on the fact that the sequence is the fixed point of the map $1 \to 121$; $2 \to 12221$, and the associated matrix of the map is

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

whose characteristic polynomial is $(X-1)(X-4)$.

Automaticity

- One can study how "close" a non-automatic sequence comes to being automatic.
- To do this, compute $(a_i)_{0 \le i \le n}$ and then form the k-kernel.
- Then (a_i) is known to $n+1$ terms, (a_{2i}) to $\lfloor n/2 \rfloor + 1$ terms, etc. Call two elements of the (partially-computed) k-kernel the same if they coincide on the terms on which they are known. The size of the k-kernel, as a function of n, is called the "automaticity" of the sequence (a_n).

Theorem. A sequence has automaticity $O(1)$ if and only if it is automatic.

Theorem. If a sequence is not automatic, then its automaticity is $\Omega_\infty(\log n)$.

Automaticity (continued)

Question. Is there a homomorphism whose fixed point is quasi-automatic, but not automatic?

Answer. Yes, the homomorphism that sends $c \to cba$; $a \to aa$; and $b \to b$ has a fixed point

$$cbabaabaaaabaaaaaaaab\cdots$$

in which the b's are in positions $2^n + n$ for $n \geq 0$. This is not a 2-automatic sequence, but it is 2-quasiautomatic. An automaton with $\leq 6\log_2 n$ states suffices to compute the sequence correctly to n terms.

Open Question. Is the fixed point of the homomorphism $1 \to 121$; $2 \to 12221$ quasi-automatic?

Automaticity (continued)

- Let $0 < \alpha < 1$ be a real irrational number with bounded partial quotients in its continued fraction expansion.

- Then it can be shown (JOS, 1995) that the automaticity of the Sturmian sequence $(s_n)_{n \geq 1}$ defined by

$$s_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$$

  is $\Omega(n^{1/5})$.

- The proof uses basic techniques of Diophantine approximation.

- In particular, can show that for any integer $r \geq 2$, and all pairs (c, d) with $c \neq d$ and $0 \leq c, d < r$, there exists an $n = O(r^3)$ such that $s_{rn+c} \neq s_{rn+d}$.

- Open Question: is the $O(r^3)$ bound best possible?

Generalization of Automatic Sequences

- Automatic sequences must take their values in a finite set
- This is too restrictive; we would like to define "automatic sequences" over the integers.
- Need the correct definition to generalize.
- Recall the <u>k-kernel</u> of a sequence:

$$K_k(a) = \{(a_{k^i n+j})_{n\geq 0} \; : \; i \geq 0, \; 0 \leq j < k^i\}.$$

- What is the proper generalization of the finiteness property?

k-regular Sequences

- An integer sequence $(a_n)_{n\geq 0}$ is said to be k-regular if the $\mathbb{Z}$-module generated by the sequences in the k-kernel is *finitely generated*.
- Example: $a_n = s_2(n)$, the total number of 1's in the binary expansion of n.
- Then $a_{2n} = a_n$ and $a_{2n+1} = a_n + 1$. It follows that $\langle K_2(a) \rangle$ is generated by $(a_n)_{n\geq 0}$ and the constant sequence 1.
- k-regular sequences appear in many different fields of mathematics: numerical analysis, topology, number theory, combinatorics, analysis of algorithms, and fractal geometry.

Examples of k-regular Sequences

Example 1. The Stern-Brocot Tree

In the limit, the sequence $(s(n))_{n\geq 0}$ of numerators one gets at level n is

$$1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, \ldots$$

which satisfies the relations $s(2n+1) = 3a(n) - a(2n)$; $s(4n) = 2a(2n) - a(n)$; $s(4n+2) = 4a(n) - a(2n)$.

Examples of k-regular Sequences

Example 2. Minimum cost of addition chains. An *addition chain* to n is a sequence of pairs of positive integers

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ ,\ldots,\ (a_r, b_r)$$

where

(i) $a_r + b_r = n$;

(ii) for all s, either $a_s = 1$, or $a_s = a_i + b_i$ for some $i < s$, and the same requirement holds for b_s.

Example: here is an addition chain to 21:

$$(1,1),\ (2,2),\ (4,1),\ (5,5),\ (10,10),\ (20,1)$$

- The *cost* of the addition chain is $\Sigma_{1 \le i \le r}\, a_i b_i$.
- Denote the cost of the minimum addition chain to n as $c(n)$.
- Graham, Yao, and Yao showed that $c(2n) = c(n) + n^2$ and $c(2n+1) = c(n) + n(n+2)$ for $n \ge 1$.
- It follows that $(c(n))_{n \ge 0}$ is 2-regular.

Examples of k-regular Sequences

Example 3. Subword Complexity. Let $w = w_0 w_1 w_2 \ldots$ be an infinite word over a finite alphabet, and let $\rho_w(n)$ be the number of distinct subwords of length n in w. Then $\rho_w(n)$ is frequently k-regular, especially when w is the fixed point of a k-uniform homomorphism. For example, when w is the Thue-Morse word $01101001\cdots$, then $\rho_w(n)$ is 2-regular.

Example 4. Mergesort. To sort a list of n integers recursively, first sort the left half (recursively), then sort the right half, and then merge the two halves together. Then $T(n)$, the total number of comparisons used in the worst case, is given by the recurrence

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n - 1.$$

It follows that $T(n)$ is 2-regular, and one can obtain the closed form

$$T(n) = n\lceil \log_2 n \rceil - 2^{\lceil \log_2 n \rceil} + 1.$$

Properties of k-regular Sequences

- Every k-automatic sequence is also k-regular.
- If a k-regular sequence is bounded, then it is k-automatic.
- The k-regular sequences are closed under shift, and periodic deletion.
- A sequence is k-regular iff it is k^r-regular for any $r \geq 2$.
- The k-regular sequences are closed under (termwise) sum and product.
- If $f(X) = \Sigma_{n \geq 0} f_n X^n$ and $g(X) = \Sigma_{n \geq 0} g_n X^n$ are formal power series with k-regular coefficients, then so is $f(X)g(X)$.
- *Conjecture:* if $(f_i)_{i \geq 0}$ and $(g_i)_{i \geq 0}$ are both k-regular sequences, and $f_i/g_i \in \mathbb{Z}$ for all $i \geq 0$, then $(f_i/g_i)_{i \geq 0}$ is also k-regular.
- *Open Question.* Show that $\lfloor \frac{1}{2} + \log_2 n \rfloor$ is not a 2-regular sequence.

The Pattern Transform

- Let $e_P(n)$ denote the number of (possibly overlapping) occurrences of the pattern P in the base-2 expansion of n. Then $e_P(n)$ is 2-regular. Furthermore, every sequence $(f_n)_{n\geq 0}$ can be expanded as a sum of such pattern sequences, and the coefficients in this sum are 2-regular if and only if $(f_n)_{n\geq 0}$ is 2-regular.

- Example:

$$\begin{aligned} e_1(3n) &= 2e_1 - 2e_{11}(n) + e_{111}(n) - 2e_{1011}(n) + \cdots \\ &= 2e_1(n) - 2\sum_{i\geq 0} e_{(10)^i 11}(n) + \sum_{i\geq 0} e_{11(01)^i 1}(n). \end{aligned}$$

- It had previously been observed by Newman that the first few values of $(e_1(3n))_{n\geq 0}$ are almost all even.

Inferring k-regular sequences

- given a sequence $(s_n)_{n \geq 0}$, how can we determine if it is k-regular?
- construct a matrix in which the rows are elements of the k-kernel, and attempt to do row reduction
- as elements further out in the k-kernel are examined, the number of columns of the matrix that are known in all entries decreases
- if rows that are previously linearly independent suddenly become dependent with the elimination of terms further out in the sequence, then no relation can be accurately deduced; stop and retry after computing more terms
- if the subsequence $(s_{k^j n+c})_{n \geq 0}$ is not linearly dependent on the previous sequences, try adding the subsequences $(s_{k^j(kn+a)+c})_{n \geq 0}$ for $0 \leq a < k$
- when no more linearly independent sequences can be found, you have found relations for the sequence

Inferring k-regular Sequences

- (N. Strauss, 1988) Define
$$r(n) = \sum_{0 \le i < n} \binom{2i}{i},$$
- let $\nu_3(n)$ be the exponent of the highest power of 3 that divides n.
- The first few terms of $\nu_3(r(n))$ are:
$$0, 1, 2, 0, 2, 3, 1, 2, 4, 0, 1, 2, 0, 3, 4, 2, 3, 5, 1, 2, \ldots$$
- A 3-regular sequence recognizer easily produces the following conjectured relations (where $f(n) = \nu_3(r(n+1))$):
- $f(3n+2) = f(n) + 2$;
- $f(9n) = f(9n+3) = f(3n)$;
- $f(9n+1) = f(9n+4) = f(9n+7) = f(3n) + 1$.
- With a little more work, one arrives at the conjecture
$$\nu_3(r(n)) = \nu_3(\binom{2n}{n}) + 2\nu_3(n).$$
- proved by Allouche and JOS.

- A beautiful proof of this identity using 3-adic analysis was also given by Don Zagier.
- Zagier showed that if we set

$$F(n) = \frac{\sum_{0\le k\le n-1} \binom{2k}{k}}{n^2\binom{2n}{n}},$$

then $F(n)$ extends to a 3-adic analytic function from $\mathbb{Z}_3$ to $-1+3\mathbb{Z}_3$, and has the expansion:

$$F(-n) = -\frac{(2n-1)!}{(n!)^2} \sum_{0\le k\le n-1} \frac{(k!)^2}{(k-1)!}.$$

<u>The Automatic Real Numbers</u>

- We say that a real number r is <u>(k, b)-automatic</u> if the base-b representation of its fractional part is a k-automatic sequence.
- For example, the number

$$.11010001000000010000000000000001\cdots_{(b)}$$

  with 1's in the 1st, 2nd, 4th, 8th, etc., positions (sometimes called the Fredholm number, although Fredholm never studied it!) is $(2, b)$-automatic.
- The set of all (k, b)-automatic numbers is denoted by $L(k, b)$.

The Automatic Reals form a Vector Space over $\mathbb{Q}$

- The proof is rather technical.
- It suffices to show that if x, y are in $L(k, b)$, then so are x/n (for any integer n) and $x + y$.
- For the first, we can express division by n as a finite-state transducer that just mimics long-division as done by hand. For example, here is a transducer for division by 3 for numbers written in base-2:

- Now a theorem of Cobham implies that $x/n \in L(k, b)$.
- For $x + y$, a more complicated proof is necessary, since carries can come from arbitrarily far to the right.
- More generally, we have a Normalization Lemma: if $(a_i)_{i \geq 1}$ is a k-automatic sequence taking values in $\mathbb{Z}$, then $\Sigma_{i \geq 1} a_i b^{-i} \in L(k, b)$.

What is the Dimension of $L(k, b)$ over $\mathbb{Q}$?

- Now that we know $L(k, b)$ is a vector space over $\mathbb{Q}$, a natural question is, what is the dimension of that vector space?
- A simple argument shows that it is infinite:
- For example, define
$$f(X) = X + X^2 + X^4 + X^8 + X^{16} + \cdots .$$
Then clearly $f(1/b^r) \in L(2, b)$ for all odd integers $r \geq 1$.
- But the numbers
$$\{f(1/b^r) \ : \ r \text{ odd}, \ \geq 1\}$$
are linearly independent over $\mathbb{Q}$.
- For if not, then we would have
$$\sum_{0 \leq i \leq s} d_i f(1/b^{2i+1}) = \sum_{0 \leq i \leq s} e_i f(1/b^{2i+1})$$
with $0 \leq d_i, e_i \leq M$ and $d_i e_i = 0$ for $0 \leq i \leq s$.
- Now for n sufficiently large, the base-b digits to the left of position $(2i+1)2^n$ on the left-hand side are $(d_i)_b$, while those in the same position on the right-hand side are $(e_i)_b$. It follows that $d_i = e_i = 0$.

Automatic Reals are Not Closed Under Product

Theorem (Lehr, Shallit, and Tromp, 1994). The automatic reals are not closed under product.

Proof. We showed that

$$f = \sum_{r \geq 0} 2^{-2^r}$$

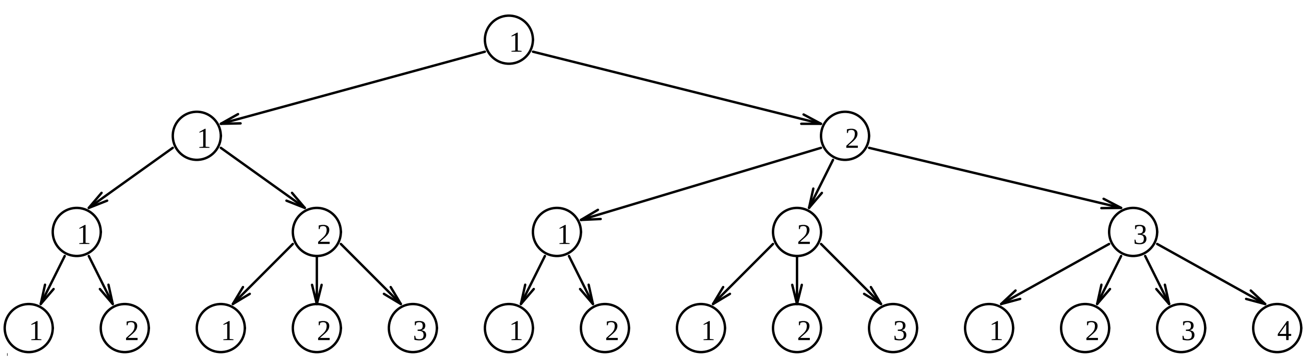
and

$$g = \sum_{m \geq 1,\ n \geq 0} 2^{-(2^m - 1)2^n}$$

are both in $L(2, 2)$, but their product is not.

Is There an Automatic Ring Strictly Containing $\mathbb{Q}$?

- Let $f(X) = X + X^2 + X^4 + X^8 + \cdots$.
- Then it can be shown that $y = f(1/b)$ is transcendental over $\mathbb{Q}$.
- Hence $\mathbb{Q}[y]$ is a ring with $[\mathbb{Q}[y] : \mathbb{Q}] = \infty$.
- If we could show all positive powers of y are in $L(k, b)$, we'd be done.
- By previous results, it suffices to show that, for any fixed r, the coefficients of $f(X)^r$ are bounded.
- Let $W_r = \max_{n \geq 0} [X^n](f(X)^r)$. Consider the following *Catalan tree*:



- Then Tromp has shown that W_r is bounded by the sum, over all vertices v at level $r-1$, of the product of all vertex labels in the path from the root to v. It follows that $W_r \leq (2r)!/(2^r \cdot r!)$.

For Further Reading

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