The Ubiquitous Thue-Morse Sequence

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Some objects in mathematics, such as

$$\pi = 3.14159\cdots$$

and

$$e = 2.71828\cdots$$

have the uncanny ability to pop up in the most unexpected places.

Ubiquity

Augustus de Morgan, in his book *A Budget of Paradoxes*, writes:

More than thirty years ago I had a friend, now long gone... One day, explaining to him how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course upon the introduction of π, which I could only describe as the ratio of the circumference of a circle to its diameter. "Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?"

The Thue-Morse Sequence

The Thue-Morse sequence

$$\mathbf{t} = (t_n)_{n \geq 0} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ \cdots$$

is another ubiquitous mathematical object.

It comes up in algebra, number theory, combinatorics, topology, and many other areas.

It has many different but equivalent definitions.

The Thue-Morse Sequence

Define a sequence of strings of 0's and 1's as follows:

$$X_0 = 0$$

$$X_{n+1} = X_n \, \overline{X_n}$$

where $\overline{x}$ means change all the 0's in x to 1's and vice-versa.

For example, we find

$$
\begin{array}{rcl}
X_0 & = & 0 \\
X_1 & = & 01 \\
X_2 & = & 0110 \\
X_3 & = & 01101001 \\
X_4 & = & 0110100110010110 \\
& \vdots &
\end{array}
$$

Then $\lim_{n \to \infty} X_n = \mathbf{t}$.

Another Definition

Given a number n we can write it in base 2,

$$n = \sum_{0 \leq i \leq k} a_i 2^i.$$

For example,

$$43 = 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0.$$

We define the "sum of digits" function $s_2(n)$ to be the sum of the a_i. So

$$s_2(43) = 1 + 0 + 1 + 0 + 1 + 1 = 4.$$

Then $t_n = s_2(n) \bmod 2$.

Let's prove this by induction. It's evidently true for $n = 0$. Now assume it is true for all $n' < n$.

- ▶ Define k by $2^k \leq n < 2^{k+1}$.
- ▶ Then t_n is the n'th symbol of $X_{k+1} = X_k \overline{X_k}$.
- ▶ So it is the $(n - 2^k)$'th symbol of $\overline{X_k}$.
- ▶ In other words, $t_n = (t_{n-2^k} + 1) \bmod 2$.
- ▶ By induction we have
  $t_{n-2^k} = s_2(n - 2^k) \bmod 2$.
- ▶ Since $2^k \leq n < 2^{k+1}$, we have $s_2(n) = s_2(n - 2^k) + 1$.
- ▶ It follows that $t_n = s_2(n) \bmod 2$.

The definition in terms of $s_2(n)$ is good because we can efficiently compute t_n without having to compute $t_0, t_1, \ldots, t_{n-1}$.

Another Definition

Here's another definition of the Thue-Morse sequence.

A morphism is a map h on strings that satisfies the identity $h(xy) = h(x)h(y)$ for all strings x, y.

Define the Thue-Morse morphism $\mu(0) = 01$, $\mu(1) = 10$. Then

$$
\begin{aligned}
\mu(0) &= 01 \\
\mu^2(0) = \mu(\mu(0)) &= 0110 \\
\mu^3(0) &= 01101001 \\
\mu^4(0) &= 0110100110010110
\end{aligned}
$$

Then $\mu^n(0) = X_n$.

Let's prove $\mu^n(0) = X_n$ by induction on n. Actually, it turns out to be easier to prove this together with the claim $\mu^n(1) = \overline{X_n}$.

These claims are clearly true for $n = 0$. Now assume they are true for n; let's prove them for $n + 1$. We have

$$
\begin{aligned}
\mu^{n+1}(0) &= \mu^n(\mu(0)) \\
&= \mu^n(01) \\
&= \mu^n(0)\mu^n(1) \\
&= X_n \, \overline{X_n} \\
&= X_{n+1}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mu^{n+1}(1) &= \mu^n(\mu(1)) \\
&= \mu^n(10) \\
&= \mu^n(1)\mu^n(0) \\
&= \overline{X_n} \, X_n \\
&= \overline{X_{n+1}}.
\end{aligned}
$$

Repetitions in Strings

A square is a string of the form xx. Examples in English include

<div align="center">

`mama`

`murmur`

`hotshots`

</div>

A word is squarefree if it contains no subword (block of consecutive symbols) that is a square. Note that `squarefree` is not squarefree, but `square` is.

A cube is a string of the form xxx. Examples in English include

<div align="center">

`hahaha`

`shshsh`

</div>

A word is cubefree if it contains no subword that is a cube.

More Repetitions in Strings

A fourth power is a string of the form *xxxx*. The only example I know of in English is

<div align="center">

`tratratratra`

</div>

which is an extinct lemur from Madagascar.

An overlap is a string of the form *axaxa* where *a* is a single letter and *x* is a string. Examples in English include

<div align="center">

`alfalfa`

`entente`

</div>

A word is overlap-free if it contains no word that is an overlap.

Theorem. *There are no squarefree strings of 0's and 1's of length ≥ 4.*

Proof. Assume x is squarefree and $|x| \geq 4$. Then without loss of generality we may assume the first symbol of x is 0. Then the second symbol must be 1, for otherwise we would have the square 00. Then the third symbol must be 0, for otherwise we would have the square 11. Thus the first three symbols are 010, and whatever symbol we choose next gives a square. Contradiction. ∎

But how about over larger alphabets?

Existence of squarefree words

Are there large squarefree words over three symbols?

A backtracking algorithm gives

$$0102012021\cdots$$

and seems to go on forever.

But how can we prove that there exists an infinite squarefree word?

This is what Thue did.

The Thue-Morse word plays a critical role.

As a first step let's show that **t** is overlap-free: it contains no subword of the form *axaxa*, with *a* a single letter and *x* a string.

Theorem

The Thue-Morse infinite word **t** *is overlap-free.*

Proof. Observe that $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$ for $n \geq 0$.

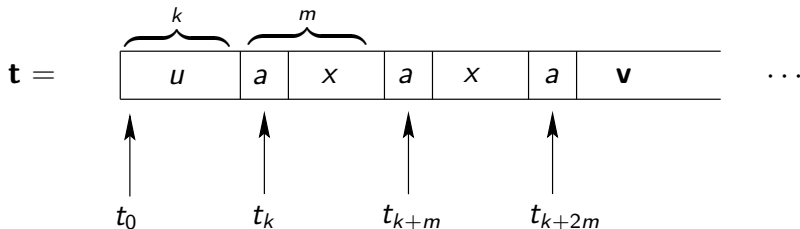Assume, contrary to what we want to prove, that **t** contains an overlap.

Figure: Hypothesized overlap in $\mathbf{t}$

Then we would be able to write $\mathbf{t} = uaxaxa\mathbf{v}$ for some finite strings u, x, an infinite string $\mathbf{v}$, and a letter a.

In other words, we would have $t_{k+j} = t_{k+j+m}$ for $0 \leq j \leq m$, where $m = |ax|$ and $k = |u|$. Assume $m \geq 1$ is as small as possible. Then there are two cases: (i) m is even; and (ii) m is odd.

(i) If m is even, then let $m = 2m'$. Again there are two cases: (a) k is even; and (b) k is odd.

(a) If k is even, then let $k = 2k'$. Then we know $t_{k+j} = t_{k+j+m}$ for $0 \le j \le m$, so it is certainly true that $t_{k+2j'} = t_{k+2j'+m}$ for $0 \le j' \le m/2$. Hence $t_{2k'+2j'} = t_{2k'+2j'+2m'}$ for $0 \le j' \le m'$, and so $t_{k'+j'} = t_{k'+j'+m'}$ for $0 \le j' \le m'$. But this contradicts the minimality of m.

(b) If k is odd, then let $k = 2k'+1$. Then as before we have $t_{k+2j'} = t_{k+2j'+m}$ for $0 \le j' \le m/2$.
Hence $t_{2k'+2j'+1} = t_{2k'+2j'+2m'+1}$ for $0 \le j' \le m'$, and so $t_{k'+j'} = t_{k'+j'+m'}$ for $0 \le j' \le m'$, again contradicting the minimality of m.

(ii) If m is odd, then there are three cases:

(a) $m \geq 5$;

(b) $m = 3$; and

(c) $m = 1$.

For $n \geq 1$, we define $b_n = (t_n + t_{n-1}) \bmod 2$.

Note that $b_{4n+2} = (t_{4n+2} + t_{4n+1}) \bmod 2$.

Since the base-2 representations of $4n+2$ and $4n+1$ are identical, except that the last two bits are switched, we have $t_{4n+2} = t_{4n+1}$, and so $b_{4n+2} = 0$.

On the other hand, $b_{2n+1} = (t_{2n+1} + t_{2n}) \bmod 2$, and the base-2 representations of $2n+1$ and $2n$ are identical except for the last bit; hence $b_{2n+1} = 1$.

(a) m odd, ≥ 5. We have $b_{k+j} = b_{k+j+m}$ for $1 \leq j \leq m$. Since $m \geq 5$, we can choose j such that $k + j \equiv 2 \pmod 4$. Then for this value of $k + j$, we have from above that $b_{k+j} = 0$, but $k + j + m$ is odd, so $b_{k+j+m} = 1$, a contradiction.

(b) $m = 3$. Again, $b_{k+j} = b_{k+j+3}$ for $1 \leq j \leq 3$. Choose j such that $k + j \equiv 2$ or $3 \pmod 4$. If $k + j \equiv 2 \pmod 4$, then the reasoning of the previous case applies. Otherwise $k + j \equiv 3 \pmod 4$, and then $b_{k+j} = 1$, while $b_{k+j+3} = 0$.

(c) $m = 1$. Then $t_k = t_{k+1} = t_{k+2}$. Hence $t_{2n} = t_{2n+1}$ for $n = \lceil k/2 \rceil$, a contradiction.

This completes the proof.

Using the fact that **t** is overlap-free, we may now construct a squarefree infinite word over the alphabet $\Sigma_3 = \{0, 1, 2\}$.

Theorem

*For $n \geq 1$, define c_n to be the number of 1's between the nth and $(n+1)$st occurrence of 0 in the word **t**. Set $\mathbf{c} = c_1 c_2 c_3 \cdots$. Then $\mathbf{c} = 210201 \cdots$ is an infinite squarefree word over the alphabet Σ_3.*

Proof. First, observe that **c** is over the alphabet $\{0, 1, 2\}$. For if there were three or more 1's between two consecutive occurrences of 0 in **t**, then **t** would not be overlap-free, a contradiction.

Next, assume that **c** is not squarefree. Then it contains a square of the form xx, with $x = x_1 x_2 \cdots x_n$ and $n \geq 1$. Then, from the definition of **c**, the word **t** would contain a subword of the form

$$01^{x_1} 01^{x_2} 0 \cdots 01^{x_n} 01^{x_1} 01^{x_2} 0 \cdots 01^{x_n} 0$$

which constitutes an overlap, a contradiction.

Thue's result is just the first step in a large and active theory: avoidability in words.

Obvious generalization: avoid fractional powers. We say x is a p/q power if x is of length p and period q. Thus outshout is an 8/5-power.

The critical exponent of a word $\mathbf{w}$ is the supremum over all α, such that $\mathbf{w}$ contains an α-power.

Recent Results in Avoidability

Some recent results:

- (Karhumäki & JOS) There are only polynomially many binary words of length n that avoid α-powers, for $\alpha \leq 7/3$;
- (Karhumäki & JOS) There are exponentially many binary words of length n that avoid $(7/3 + \epsilon)$-powers.
- (Rampersad) The Thue-Morse word and its complement are the only words that are the fixed points of a non-trivial morphism and avoid α-powers for $2 < \alpha \leq 7/3$.
- (Brown, Rampersad, Vasiga, & JOS) If you change any finite number of bits in the Thue-Morse word, it has an overlap.
- (Krieger) The critical exponent for uniform binary morphisms is rational, and for non-erasing morphisms is algebraic.
- (Kreiger & JOS) For each real $\alpha > 1$ there exists an infinite word with α as critical exponent.

Consider the sequence

$$\frac{1}{2}, \quad \frac{1/2}{3/4}, \quad \frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}, \quad \frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}} \Big/ \frac{\frac{9/10}{11/12}}{\frac{13/14}{15/16}}, \dots$$

What does this converge to?

An Amazing Infinite Product

We find

$$\frac{1}{2} \doteq .500$$

$$\frac{1/2}{3/4} = 2/3 \doteq .666$$

$$\frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}} = 7/10 \doteq .700$$

Numerically, the limit seems to be $0.7071\cdots$.

Let's prove that this sequence converges to $\frac{\sqrt{2}}{2}$.

An Amazing Infinite Product

First, we observe that the limit is

$$\prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(-1)^{t_n}} \tag{1}$$

where t_n is the sum of the bits (mod 2) in the binary expansion of n.

An Amazing Infinite Product

We now use a trick of Allouche: let

$$P = \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(-1)^{t_n}}$$

and define

$$Q = \prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{(-1)^{t_n}}.$$

Clearly

$$PQ = \frac{1}{2} \prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{(-1)^{t_n}}.$$

Now break this infinite product into separate products over odd
and even indices; we find

$$
\begin{aligned}
PQ &= \frac{1}{2} \prod_{n \geq 1} \left(\frac{n}{n+1} \right)^{(-1)^{t_n}} \\
&= \frac{1}{2} \prod_{n \geq 0} \left(\frac{2n+1}{2n+2} \right)^{(-1)^{t_{2n+1}}} \prod_{n \geq 1} \left(\frac{2n}{2n+1} \right)^{(-1)^{t_n}} \\
&= \frac{1}{2} P^{-1} Q.
\end{aligned}
$$

It follows that $P^2 = \frac{1}{2}$.

But how about Q? Is it irrational? Transcendental?

I offer \$25 for the answewr to this question.

The Multigrades Problem

The multigrades problem is the following: let I and J be disjoint sets. Can one find "short" solutions to the system of equations

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k$$

for $k = 0, 1, 2, \ldots, t$?

For example, one solution for $t = 2$ is the identity

$$0^k + 3^k + 5^k + 6^k = 1^k + 2^k + 4^k + 7^k$$

for $k = 0, 1, 2$.

In 1851, the French mathematician Étienne Prouhet gave the following general solution.

The Multigrades Problem

Theorem

The Thue-Morse sequence $\mathbf{t} = (t_n)_{n \geq 0}$ *has the following property.*
Define

$$I = \{0 \leq i < 2^N \ : \ t_i = 0\}$$

$$J = \{0 \leq j < 2^N \ : \ t_j = 1\}$$

Then for $0 \leq k < N$ *we have*

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k.$$

For example, for $N = 3$ we have the partition obtained before:

$$0^k + 3^k + 5^k + 6^k = 1^k + 2^k + 4^k + 7^k$$

for $k = 0, 1, 2$.

Proof.

We actually prove a more general theorem by induction on N. We prove that if p is any polynomial of degree $< N$, then

$$\sum_{\substack{0 \leq i < 2^N \\ t_i = 0}} p(i) = \sum_{\substack{0 \leq j < 2^N \\ t_j = 1}} p(j)$$

The desired result then follows by successively considering the case $p(i) = 1$, $p(i) = i$, $p(i) = i^2$, etc.

The base case is $N = 1$. Then p is a constant, the result clearly follows.

The Multigrades Problem

Proof.

Now assume the result true for all polynomials of degree $< N$.
We try to prove it for a polynomial $p(x)$ of degree N.
Consider the polynomial $p(x + 2^N) - p(x)$.
If

$$p(x) = a_N x^N + a_{N-1} x^{N-1} + \cdots + a_1 x + a_0,$$

then $p(x + 2^N) = a_N(x + 2^N)^N +$ smaller degree terms, which by the binomial theorem, is $a_N(x^N +$ smaller degree terms).
So $p(x + 2^N) - p(x)$ is actually a polynomial of degree $< N$. So we can apply induction to it. We get

The Multigrades Problem

$$\sum_{\substack{0 \leq i < 2^N \\ t_i = 0}} \left(p(i + 2^N) - p(i) \right) = \sum_{\substack{0 \leq j < 2^N \\ t_j = 1}} \left(p(j + 2^N) - p(j) \right)$$

So, rearranging, we get

$$\sum_{\substack{0 \leq i < 2^N \\ t_i = 0}} p(i + 2^N) + \sum_{\substack{0 \leq j < 2^N \\ t_j = 1}} p(j)$$

$$= \sum_{\substack{0 \leq j < 2^N \\ t_j = 1}} p(j + 2^N) + \sum_{\substack{0 \leq i < 2^N \\ t_i = 0}} p(i).$$

Hence

$$\sum_{\substack{0 \leq i < 2^{N+1} \\ t_i = 0}} p(i) = \sum_{\substack{0 \leq j < 2^{N+1} \\ t_j = 1}} p(j).$$

We're done.

An Exercise

Exercise: find the appropriate generalization for bases larger than 2.

An Open Problem

The Thue-Morse partition of $\{0, 1, \ldots, 2^N - 1\}$ is not the unique partition such that

$$\sum_{i \in I} i^k = \sum_{j \in J} j^k$$

for $k = 0, 1, \ldots, N - 1$.

But is it the partition that minimizes $\left| \sum_{i \in I} i^N - \sum_{j \in J} j^N \right|$?

And if so, is it unique?

Yet another definition of the Thue-Morse sequence **t** can be given in terms of power series. Let X be an indeterminate. We have

$$
\begin{aligned}
\prod_{i \geq 0}(1 - X^{2^i}) &= (1 - X)(1 - X^2)(1 - X^4)\cdots \\[2mm]
&= 1 - X - X^2 + X^3 - X^4 \cdots \\[2mm]
&= \sum_{j \geq 0}(-1)^{t_j} X^j.
\end{aligned}
$$

Another Definition

Something even more interesting arises when we consider Laurent series over $GF(2)$, the finite field with two elements. Basically, we do all arithmetic operations as usual, but reduce modulo 2.

For example, consider the Laurent series

$$G(X) = X^{-1} + X^{-2} + X^{-4} + X^{-8} + \cdots.$$

It turns out that this series is algebraic over $GF(2)(X)$. By this we mean that G is the analogue of an algebraic number, a number satisfying an algebraic equation.

Formal Power Series

Let's try to find the equation that

$$G(X) = X^{-1} + X^{-2} + X^{-4} + X^{-8} + \cdots$$

satisfies.

What is $G(X)^2$? If we compute it over the integers, we get

$$X^{-2} + 2X^{-3} + X^{-4} + 2X^{-5} + 2X^{-6} + X^{-8} + 2X^{-9} + \cdots .$$

Reduced mod 2, this is just

$$X^{-2} + X^{-4} + X^{-8} + X^{-16} + \cdots .$$

More generally, if we have a power series $H(X)$, then
$H(X)^p = H(X^p)$ over $GF(p)$, where p is a prime number.
To see this, it suffices to remember that

$$(a + b)^p \equiv a^p + b^p \pmod{p}.$$

For $G(X) = X^{-1} + X^{-2} + X^{-4} + X^{-8} + \cdots$ we get
$G(X)^2 = G(X) - X^{-1}$, and so

$$G^2 + G + X^{-1} = 0.$$

Thus G is quadratic.

The Thue-Morse Power Series

Theorem
Let $F(X) = \sum_{n \geq 0} t_n X^{-n}$. Then, over $GF(2)$, the Laurent series F satisfies a quadratic equation with coefficients that are polynomials in X. More precisely, we have

$$(1 + X)^3 F^2 + X(1 + X)^2 F + X^2 = 0. \qquad (2)$$

The Thue-Morse Power Series

Proof.

$$
\begin{aligned}
F &= \sum_{n \geq 0} t_n X^{-n} \\
&= \sum_{n \geq 0} t_{2n} X^{-2n} + \sum_{n \geq 0} t_{2n+1} X^{-2n-1} \\
&= \sum_{n \geq 0} t_n X^{-2n} + X^{-1} \sum_{n \geq 0} (1 + t_n) X^{-2n} \\
&= F^2 + X^{-1} \left(\frac{X^2}{1 + X^2} + F^2 \right) \\
&= \left(\frac{1 + X}{X} \right) F^2 + \frac{X}{1 + X^2} \\
&= \left(\frac{1 + X}{X} \right) F^2 + \frac{X}{(1 + X)^2}.
\end{aligned}
$$

We have

$$F = \left(\frac{1+X}{X}\right) F^2 + \frac{X}{(1+X)^2}.$$

Hence, multiplying through by $X(1+X)^2$, we obtain

$$(1+X)^3 F^2 + X(1+X)^2 F + X^2 = 0.$$

The fact that F is not a rational function is an easy consequence of the overlap-free property of the sequence $\mathbf{t}$.

Recall that an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}$$

is called a continued fraction.
It is usually abbreviated as $[a_0, a_1, a_2, \ldots, a_n]$.
We can also consider infinite expressions of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

which is abbreviated $[a_0, a_1, a_2, \ldots]$.
The terms a_i are positive integers, except for a_0, which may be any integer.

A continued fraction where the terms include 0 can be converted into the ordinary form using the identity

$$[\ldots, a, 0, b, \ldots] = [\ldots, a + b, \ldots].$$

Now consider the continued fraction where the terms are the Thue-Morse sequence:

$$[0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots].$$

Using the collapsing rule gives

$$[0, 1, 2, 1, 1, 2, \ldots].$$

This number is transcendental.

The terms of this continued fraction, disregarding the initial 0, form the run-lengths of the symbols in the Thue-Morse sequence.

Characterize all infinite sequences of 0's and 1's such that when 0's are collapsed, the result is the run lengths of the original sequence, shifted by one.

According to official rule (10.12) of the game of chess, a player can claim a draw if "at least 50 consecutive moves have been made by each side without the capture of any piece and without the movement of any pawn". Actually, this is not enough for certain positions, such as King + Rook + Bishop versus King + 2 Knights, so the rule also stipulates that "This number of 50 moves can be increased for certain positions, provided that this increase in number and these positions have been clearly announced by the organisers before the event starts."

Another rule (10.10) allows a draw to be claimed if the same position occurs for the third time. By "same position" we mean that the pieces are in the same position, including the rights to castle or capture a pawn *en passant*. Without these two rules, infinite games are clearly possible. However, can rule (10.10) be weakened and still disallow infinite chess games?

The Thue-Morse Sequence and Chess

Consider the following alternative rule: a draw occurs if the same sequence of moves occurs twice in succession and is immediately followed by the first move of a third repetition.
Can an infinite game of chess occur under this rule?

The question was answered by Max Euwe, the Dutch chess master (and world champion from 1935–1937) in 1929.

Figure: Max Euwe (1901–1981)

The Thue-Morse Sequence and Chess

Euwe's construction used the Thue-Morse sequence! (He discovered it independently.)

One way to do this is to take the Thue-Morse sequence and map 0 to a sequence of moves, and 1 to another sequence of moves. For example, one way is as follows:

$$
\begin{array}{rl}
0 \quad \rightarrow \quad & \text{Ng1} - \text{f3} \qquad \text{Ng8} - \text{f6} \\
& \text{Nf3} - \text{g1} \qquad \text{Nf6} - \text{g8}
\end{array}
$$

$$
\begin{array}{rl}
1 \quad \rightarrow \quad & \text{Nb1} - \text{c3} \qquad \text{Nb8} - \text{c6} \\
& \text{Nc3} - \text{b1} \qquad \text{Nc6} - \text{b8}
\end{array}
$$

The Thue-Morse Sequence and Music

The Thue-Morse sequence has even been used in composing music!

Tom Johnson, a Paris-based composer, has used the Thue-Morse
sequence and other sequences formed by iterated morphisms, in his
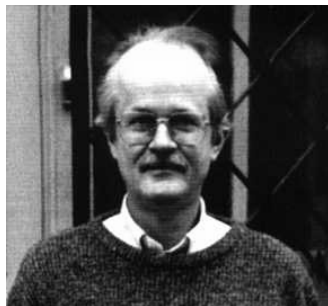work.

Figure: Tom Johnson

Figure: Composition by Tom Johnson

Some compositions based on the "infinity series"

$$0, 1, -1, 2, 1, 0, -2, 3, -1, 2, 0, 1, 2, -1, -3, 4, 1, \ldots$$

defined by

$$
\begin{aligned}
c_0 &= 0 \\
c_{2n} &= -c_n \\
c_{2n+1} &= c_n + 1.
\end{aligned}
$$

Note that $c_n \equiv t_n \pmod 2$.

For Further Reading

Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.

Jean-Paul Allouche and Jeffrey Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in C. Ding, T. Helleseth, and H. Niederreiter, eds., *Sequences and Their Applications: Proceedings of SETA '98*, Springer-Verlag, 1999, pp. 1-16. Also available at
`http://www.cs.uwaterloo.ca/~shallit/papers.html`

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`http://tom.johnson.org`

`http://www.pernoergaard.dk`