# The Frobenius Problem and Its Generalizations

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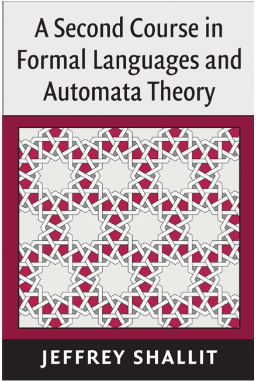


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The **Frobenius problem** is the following: given positive integers $x_1, x_2, \ldots, x_n$ with $\gcd(x_1, x_2, \ldots, x_n) = 1$, compute the largest integer **not** representable as a non-negative integer linear combination of the $x_i$.

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The restriction $\gcd(x_1, x_2, \ldots, x_n) = 1$ is necessary for the definition to be meaningful, for otherwise every non-negative integer linear combination is divisible by this gcd.

# The Chicken McNuggets Problem

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*At McDonald's, Chicken McNuggets are available in packs of either 6, 9, or 20 nuggets. What is the largest number of McNuggets that one cannot purchase?*

# The Chicken McNuggets Problem

Answer: 43.

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To see that 43 is not representable, observe that we can choose either 0, 1, or 2 packs of 20. If we choose 0 or 1 packs, then we have to represent 43 or 23 as a linear combination of 6 and 9, which is impossible. So we have to choose two packs of 20. But then we cannot get 43.

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48 &= 0 \cdot 20 + 0 \cdot 9 + 8 \cdot 6 \\
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\end{aligned}
$$

and every larger number can be written as a multiple of 6 plus one of these numbers.

# History of the Frobenius problem

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- Applications of the Frobenius problem occur in number theory, automata theory, sorting algorithms, etc.

# Research on the Frobenius problem

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- Complexity of computing $g$

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So $xy - x - y$ is not representable.

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For example, for $[x, y] = [13, 19]$, we find $[2, 10] \cdot [x, y] = 216$. Also $[3, -2] \cdot [x, y] = 1$. To get a representation for 217, we just add these two vectors to get $[5, 8]$.

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Kannan has given a polynomial-time algorithm for any fixed dimension, but the time depends at least exponentially on the dimension and the algorithm is very complicated.

# Computational Complexity of $g$

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The **integer knapsack problem** is, given $x_1, x_2, \ldots, x_n$, and a target $t$, do there exist non-negative integers $a_i$ such that $\sum_{1 \leq i \leq n} a_i x_i = t$ ?

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His reduction requires 3 calls to a subroutine for the Frobenius number $g$.

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# Applications of the Frobenius Number

- Shell sort - a sorting algorithm devised by D. Shell in 1959.

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- Basic idea: arrange list in $j$ columns; sort columns; decrease $j$; repeat

# Shellsort Example

Start with 10 5 12 13 4 6 9 11 8 1 7

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Arrange in 5 columns:

| | | | | |
|---|---|---|---|---|
| 10 | 5 | 12 | 13 | 4 |
| 6 | 9 | 11 | 8 | 1 |
| 7 | | | | |

# Shellsort Example

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Arrange in 5 columns:

| | | | | |
|---|---|---|---|---|
| 10 | 5 | 12 | 13 | 4 |
| 6 | 9 | 11 | 8 | 1 |
| 7 | | | | |

Sort each column:

| | | | | |
|---|---|---|---|---|
| 6 | 5 | 11 | 8 | 1 |
| 7 | 9 | 12 | 13 | 4 |
| 10 | | | | |

# Shellsort Example

Now arrange in 3 columns:

| | | |
|---|---|---|
| 6 | 5 | 11 |
| 8 | 1 | 7 |
| 9 | 12 | 13 |
| 4 | 10 | |

# Shellsort Example

Now arrange in 3 columns:

| | | |
|---|---|---|
| 6 | 5 | 11 |
| 8 | 1 | 7 |
| 9 | 12 | 13 |
| 4 | 10 | |

Sort each column:

| | | |
|---|---|---|
| 4 | 1 | 7 |
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# Shellsort Example

We now have
4 1 7 6 5 11 8 10 13 9 12.

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Finally, sort the remaining elements:
1 4 5 6 7 8 9 10 11 12 13

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- It is $O(n(\log n)^2)$ if increments $1, 2, 3, 4, 6, 9, 8, 12, 18, 27, 16, 24, \ldots$ are used (Numbers of the form $2^i 3^j$).

# Shellsort and the Frobenius Problem

**Theorem.** The number of steps required to $r$-sort a file $a[1..N]$ that is already $r_1, r_2, \ldots, r_t$-sorted is $\leq \frac{N}{r} g(r_1, r_2, \ldots, r_t)$.

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*Proof.* The number of steps to insert $a[i]$ is the number of elements in $a[i-r], a[i-2r], \ldots$ that are greater than $a[i]$. But if $x$ is a linear combination of $r_1, r_2, \ldots, r_t$, then $a[i-x] < a[i]$, since the file is $r_1, r_2, \ldots, r_t$-sorted.

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# The Frobenius Problem and NFA to DFA Conversion

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However, for unary languages, the $2^n$ bound is not attainable.

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Chrobak showed that any unary $n$-state NFA can be put into a certain normal form, where there is a "tail" of $< n^2$ states, followed by a single nondeterministic state which has branches into different cycles, where the total number of states in all the cycles is $\leq n$.

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The bound of $n^2$ for the number of states in the tail comes from the bound we have already seen on the Frobenius problem.

# An Exercise

Use the Frobenius problem on two variables to show that the language

$$L_n = \{\mathtt{a}^i \ : \ i \neq n\}$$

can be accepted by an NFA with $O(\sqrt{n})$ states.

# Related Problems

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He also gave the formula $h(x_1, x_2) = \frac{1}{2}(x_1 - 1)(x_2 - 1)$.

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# Related Problems

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**Theorem.** $h(x_1, x_2, \ldots, x_k) = h(x_1, x_2, \ldots, x_k, d)$ if and only iff $d$ can be expressed as a non-negative integer linear combination of the $x_i$.

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It follows that the integer knapsack problem (known to be NP-complete) can be reduced to the problem of computing $h$, and so computing $h$ is also NP-hard (under Turing reductions).

# The Local Postage Stamp Problem

In this problem, we are given a set of denominations $1 = x_1, x_2, \ldots, x_k$ of stamps, and an envelope that can contain at most $t$ stamps. We want to determine the *smallest* amount of postage we *cannot* provide. Call it $N_t(x_1, x_2, \ldots, x_k)$.

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Alter and Barnett asked (1980) if $N_t(x_1, x_2, \ldots, x_k)$ can be "expressed by a simple formula".

The answer is, probably not. I proved computing $N_t(x_1, x_2, \ldots, x_k)$ is NP-hard in 2001.

# The Global Postage-Stamp Problem

The global postage-stamp problem is yet another variant: now we are given a limit $t$ on the number of stamps to be used, and an integer $k$, and the goal is to find a set of $k$ denominations $x_1, x_2, \ldots, x_k$ that maximizes $N_t(x_1, x_2, \ldots, x_k)$.

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The complexity of this problem is unknown.

# The Optimal Coin Change Problem

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It turns out that the system of denominations $(1, 5, 18, 25)$ is optimal, with an average cost of only 3.89 coins per amount.

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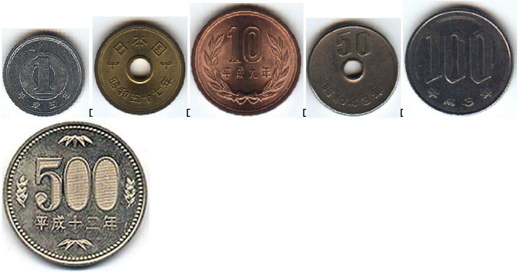


Japan uses a system based on 1 and 5: there are coins of 1 yen, 5 yen, 10 yen, 50 yen, 100 yen, and 500 yen. But switching to 1 and 3 (or 1 and 4) would decrease the average number of coins used.

# Generalizing the Frobenius Problem to Words

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We can now replace the integers $x_i$ with words (strings of symbols over a finite alphabet $\Sigma$), and ask, what is the right generalization of the Frobenius problem?

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Instead of the condition that $\gcd(x_1, x_2, \ldots, x_k) = 1$, which was used to ensure that the number of unrepresentable integers is finite, we could demand that

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be *co-finite*.

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**Theorem.** Let $x_1, x_2, \ldots, x_k \in \Sigma^+$. Then $x_1^* x_2^* \cdots x_k^*$ is co-finite if and only if $|\Sigma| = 1$ and $\gcd(|x_1|, \ldots, |x_k|) = 1$.

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For the other direction, suppose $Q$ is co-finite. If $|\Sigma| = 1$, let $\gcd(|x_1|, \ldots, |x_k|) = d$. If $d > 1$, $Q$ contains only words of length divisible by $d$, and so is not co-finite. So $d = 1$.

Hence assume $|\Sigma| \geq 2$, and let $a, b$ be distinct letters in $\Sigma$.

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For if none of the $x_i$ consists of powers of a single letter, then the longest block of consecutive identical letters in any word in $Q$ is $< 2\ell$, so no word in $Q'$ can be in $Q$.

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Take any word $w$ in $Q$, and count the number $n(w)$ of maximal blocks of $2\ell$ or more consecutive identical letters in $w$. (Here "maximal" means such a block is delimited on both sides by either the beginning or end of the word, or a different letter.)

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Clearly $n(w) \leq k$.

But $n(w') \geq 2k$ for any word $w'$ in $Q'$. Thus $Q$ is not co-finite, as it omits all the words in $Q'$. □

## $\{x_1, x_2, \ldots, x_k\}^*$

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Since this DFA accepts a finite language, the longest word it accepts is bounded by the number of states.

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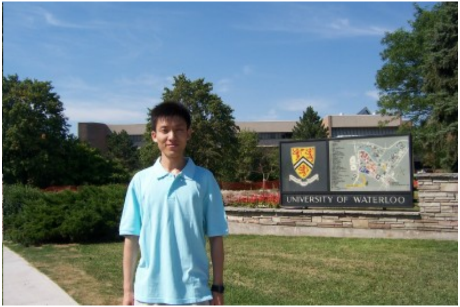


My student Zhi Xu has recently produced a class of examples $\{x_1, x_2, \ldots, x_k\}$ in which the length of the longest word is $n$, but the longest word in $\Sigma^* - \{x_1, x_2, \ldots, x_k\}^*$ is exponential in $n$.

# $\{x_1, x_2, \ldots, x_k\}^*$: Zhi Xu's Examples

Let $r(n, k, l)$ denote the word of length $l$ representing $n$ in base $k$, possibly with leading zeros. For example, $r(3, 2, 3) = 011$.

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**Example.** Let $m = 3, n = 5, \Sigma = \{0, 1\}$. In this case, $l = 3 \cdot 2^2 + 2 = 14$, $S = \Sigma^3 + \Sigma^5 - \{00001, 01010, 10011\}$. Then a longest word not in $S^*$ is

$$00001010011\ 000\ 00001010011$$

of length $25 = g(3, 14)$.

# Counting the Omitted Words

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# Other Possible Generalizations

Instead of considering the longest word omitted by $x_1^* x_2^* \cdots x_k^*$ or $\{x_1, x_2, \ldots, x_k\}^*$, we might consider their state complexity.

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It turns out that the state complexity of $\{x_1, x_2, \ldots, x_k\}^*$ can be exponential in both the length of the longest word and the number of words.

**Theorem.** Let $t$ be an integer $\geq 2$, and define words as follows:

$$y := 01^{t-1}0$$

and

$$x_i := 1^{t-i-1}01^{i+1}$$

for $0 \leq i \leq t-2$. Let $S_t := \{0, x_0, x_1, \ldots, x_{t-2}, y\}$. Then $S_t^*$ has state complexity $3t2^{t-2} + 2^{t-1}$.

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**Example.** For $t = 6$ the words in $S_t$ are 0 and

$$\begin{aligned} y &= 0111110 \\ x_0 &= 1111101 \\ x_1 &= 1111011 \\ x_2 &= 1110111 \\ x_3 &= 1101111 \\ x_4 &= 1011111 \end{aligned}$$

# State Complexity

Using similar ideas, we can also create an example achieving subexponential state complexity for $x_1^* x_2^* \cdots x_k^*$.

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**Theorem.** Let $y$ and $x_i$ be as defined above. Let $L = (0^* x_1^* x_2^* \cdots x_{n-1}^* y^*)^e$ where $e = (t+1)(t-2)/2 + 2t$. Then $\mathrm{sc}(L) \geq 2^{t-2}$.

This example is due to Jui-Yi Kao.

**Theorem.** If $S$, a finite list of words, is represented by either an NFA or a regular expression, then determining if $S^*$ is co-finite is NP-hard and is in PSPACE.

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**Theorem.** If $S$ is a unary language (possibly infinite) represented by an NFA, then we can decide in polynomial time if $S^*$ is co-finite.

# Open Problem

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Given a finite list of words $S = \{x_1, x_2, \ldots, x_k\}$, determine if $S^*$ is co-finite.

# For Further Reading


- J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, 2005.
- J. Shallit, The computational complexity of the local postage stamp problem, *SIGACT News* **33** (1) (March 2002), 90–94.
- J. Shallit, What this country needs is an 18-cent piece, *Math. Intelligencer* **25** (2) (2003), 20–23.
- Jui-Yi Kao, J. Shallit, and Zhi Xu, "The Frobenius problem in a free monoid", in S. Albers and P. Weil, eds., *STACS 2008, 25th Annual Symposium on Theoretical Aspects of Computer Science*, 2008, pp. 421–432.