The Frobenius Problem and Its Generalizations

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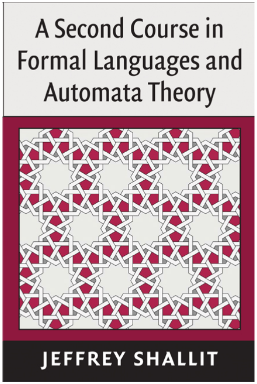

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The Frobenius Problem

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The restriction $\gcd(x_1, x_2, \ldots, x_n) = 1$ is necessary for the definition to be meaningful, for otherwise every non-negative integer linear combination is divisible by this gcd.

The Chicken McNuggets Problem

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At McDonald's, Chicken McNuggets are available in packs of either 6, 9, or 20 nuggets. What is the largest number of McNuggets that one cannot purchase?

The Chicken McNuggets Problem

Answer: 43.

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To see that 43 is not representable, observe that we can choose either 0, 1, or 2 packs of 20. If we choose 0 or 1 packs, then we have to represent 43 or 23 as a linear combination of 6 and 9, which is impossible. So we have to choose two packs of 20. But then we cannot get 43.

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48 &= 0 \cdot 20 + 0 \cdot 9 + 8 \cdot 6 \\
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\end{aligned}$$

and every larger number can be written as a multiple of 6 plus one of these numbers.

History of the Frobenius problem

- Problem discussed by Frobenius (1849–1917) in his lectures in the late 1800's — but Frobenius never published anything

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- Applications of the Frobenius problem occur in number theory, automata theory, sorting algorithms, etc.

Research on the Frobenius problem

- Formulas for g where dimension is bounded

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So $xy - x - y$ is not representable.

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For example, for $(x, y) = [13, 19]$, we find $[2, 10] \cdot [x, y] = 216$. Also $[3, -2] \cdot [x, y] = 1$. To get a representation for 217, we just add these two vectors to get $[5, 8]$.

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Kannan has given a polynomial-time algorithm for any fixed dimension, but the time depends at least exponentially on the dimension and the algorithm is very complicated.

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His reduction requires 3 calls to a subroutine for the Frobenius number g.

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Applications of the Frobenius Number

- Shell sort - a sorting algorithm devised by D. Shell in 1959.

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- Basic idea: arrange list in j columns; sort columns; decrease j; repeat

Shellsort Example

Start with 10 5 12 13 4 6 9 11 8 1 7

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Arrange in 5 columns:

10	5	12	13	4
6	9	11	8	1
7				

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Arrange in 5 columns:

```
10   5   12   13   4
 6   9   11    8   1
 7
```

Sort each column:

```
 6   5   11    8   1
 7   9   12   13   4
10
```

Shellsort Example

Now arrange in 3 columns:

6	5	11
8	1	7
9	12	13
4	10	

Shellsort Example

Now arrange in 3 columns:

6	5	11
8	1	7
9	12	13
4	10	

Sort each column:

4	1	7
6	5	11
8	10	13
9	12	

Shellsort Example

Finally, sort the remaining elements:
1 4 5 6 7 8 9 10 11 12 13

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- It is $O(n(\log n)^2)$ if increments $1, 2, 3, 4, 6, 9, 8, 12, 18, 27, 16, 24, \ldots$ are used (Numbers of the form $2^i 3^j$).

Shellsort and the Frobenius Problem

Theorem. The number of steps required to r-sort a file $a[1..N]$ that is already $r_1, r_2, \ldots, r_t$-sorted is $\leq \frac{N}{r} g(r_1, r_2, \ldots, r_t)$.

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The Frobenius Problem and NFA to DFA Conversion

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However, for unary languages, the 2^n bound is not attainable.

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The bound of n^2 for the number of states in the tail comes from the bound we have already seen on the Frobenius problem.

An Exercise

Use the Frobenius problem on two variables to show that the language

$$L_n = \{\mathtt{a}^i \ : \ i \neq n\}$$

can be accepted by an NFA with $O(\sqrt{n})$ states.

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There is a very simple proof of this formula. Consider all the numbers between 0 and $(x_1 - 1)(x_2 - 1)$. Then it is not hard to see that every representable number in this range is paired with a non-representable number via the map $c \to c'$, where $c' = (x_1 - 1)(x_2 - 1) - c - 1$, and vice-versa.

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However, the complexity of computing h is apparently still open (but see next slide)

Computing h is NP-hard

After my talk at DLT '08, Pawel Gawrychowski pointed out the following very simple argument that computing h is NP-hard:

Theorem. $h(a_1, a_2, \ldots, a_k) = h(a_1, a_2, \ldots, a_k, d)$ if and only iff d can be expressed as a non-negative integer linear combination of the a_i.

It follows that the integer knapsack problem (known to be NP-complete) can be reduced to the problem of computing h, and so computing h is also NP-hard.

The Local Postage Stamp Problem

In this problem, we are given a set of denominations $1 = x_1, x_2, \ldots, x_k$ of stamps, and an envelope that can contain at most t stamps. We want to determine the *smallest* amount of postage we *cannot* provide. Call it $N_t(x_1, x_2, \ldots, x_k)$.

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The answer is, probably not. I proved computing $N_t(x_1, x_2, \ldots, x_k)$ is NP-hard in 2001.

The Global Postage-Stamp Problem

The global postage-stamp problem is yet another variant: now we are given a limit t on the number of stamps to be used, and an integer k, and the goal is to find a set of k denominations $x_1, x_2, \ldots, x_k$ that maximizes $N_t(x_1, x_2, \ldots, x_k)$.

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The complexity of this problem is unknown.

The Optimal Coin Change Problem

Yet another variant is the optimal change problem: here we are given a bound on the number of distinct coin denominations we can use (but allowing arbitrarily many of each denomination), and we want to find a set that minimizes the average number of coins needed to make each amount in some range.

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It turns out that the system of denominations $(1, 5, 18, 25)$ is optimal, with an average cost of only 3.89 coins per amount.

Improving the Current Coin System

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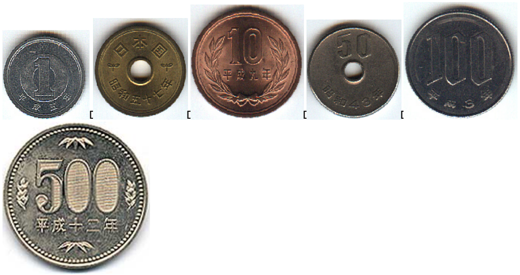
Improving the Japanese System



Japan uses a system based on 1 and 5: there are coins of 1 yen, 5 yen, 10 yen, 50 yen, 100 yen, and 500 yen. But switching to 1 and 3 (or 1 and 4) would decrease the average number of coins used.

Generalizing the Frobenius Problem to Words

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We can now replace the integers x_i with words (strings of symbols over a finite alphabet Σ), and ask, what is the right generalization of the Frobenius problem?

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or

$$\{x_1, x_2, \ldots, x_k\}^*.$$

Generalizing the Frobenius Problem to Words

Instead of the condition that $\gcd(x_1, x_2, \ldots, x_k) = 1$, which was used to ensure that there the number of unrepresentable integers is finite, we could demand that

$$\Sigma^* - x_1^* x_2^* \cdots x_k^*$$

or

$$\Sigma^* - \{x_1, x_2, \ldots, x_k\}^*$$

be finite, or in other words, that

$$x_1^* x_2^* \cdots x_k^*$$

or

$$\{x_1, x_2, \ldots, x_k\}^*$$

be *co-finite*.

Generalizing the Frobenius Problem to Words

And instead of looking for the largest non-representable integer, we could ask for the **length of the longest word** not in

$$x_1^* x_2^* \cdots x_k^*$$

or

$$\{x_1, x_2, \ldots, x_k\}^*.$$

$x_1^* x_2^* \cdots x_k^*$

Theorem. Let $x_1, x_2, \ldots, x_k \in \Sigma^+$. Then $x_1^* x_2^* \cdots x_k^*$ is co-finite if and only if $|\Sigma| = 1$ and $\gcd(|x_1|, \ldots, |x_k|) = 1$.

Proof. Let $Q = x_1^* x_2^* \cdots x_k^*$.

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If $|\Sigma| = 1$ and $\gcd(|x_1|, \ldots, |x_k|) = 1$, then every sufficiently long unary word can be obtained by concatenations of the x_i, so Q is co-finite.

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For the other direction, suppose Q is co-finite. If $|\Sigma| = 1$, let $\gcd(|x_1|, \ldots, |x_k|) = d$. If $d > 1$, Q contains only words of length divisible by d, and so is not co-finite. So $d = 1$.

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Let $Q' = ((a^{2\ell} b^{2\ell})^k)^+$. Then we claim that $Q' \cap Q = \emptyset$.

For if none of the x_i consists of powers of a single letter, then the longest block of consecutive identical letters in any word in Q is $< 2\ell$, so no word in Q' can be in Q.

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Clearly $n(w) \leq k$.

But $n(w') \geq 2k$ for any word w' in Q'. Thus Q is not co-finite, as it omits all the words in Q'. □

$\{x_1, x_2, \ldots, x_k\}^*$

Suppose $\max_{1 \le i \le k} |x_i| = n$.

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Since this DFA accepts a finite language, the longest word it accepts is bounded by the number of states.

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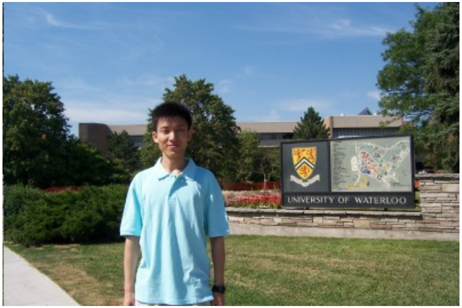

My student Zhi Xu has recently produced a class of examples $\{x_1, x_2, \ldots, x_k\}$ in which the length of the longest word is n, but the longest word in $\Sigma^* - \{x_1, x_2, \ldots, x_k\}^*$ is exponential in n.

$\{x_1, x_2, \ldots, x_k\}^*$: Zhi Xu's Examples

Let $r(n, k, l)$ denote the word of length l representing n in base k, possibly with leading zeros. For example, $r(3, 2, 3) = 011$.

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Let $T(m, n) = \{r(i, |\Sigma|, n - m)0^{2m-n}r(i + 1, |\Sigma|, n - m) \;:\; 0 \leq i \leq |\Sigma|^{n-m} - 2\}$.

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Theorem. Let m, n be integers with $0 < m < n < 2m$ and $\gcd(m,n) = 1$, and let $S = \Sigma^m + \Sigma^n - T(m,n)$. Then S^* is co-finite and the longest words not in S^* are of length $g(m,l)$, where $l = m|\Sigma|^{n-m} + n - m$.

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Example. Let $m = 3, n = 5, \Sigma = \{0, 1\}$. In this case, $l = 3 \cdot 2^2 + 2 = 14$, $S = \Sigma^3 + \Sigma^5 - \{00001, 01010, 10011\}$. Then a longest word not in S^* is

$$00001010011\ 000\ 00001010011$$

of length $25 = g(3, 14)$.

Counting the Omitted Words

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Example. Let $m = 3, n = 5, \Sigma = \{0,1\}$. Then $S = \Sigma^3 + \Sigma^5 - \{00001, 00010, 00011, 01010, 01011, 10011\}$. Then S^* omits $1712 > 11 = 2^{2^2} - 2^2 - 1$ words.

Other Possible Generalizations

Instead of considering the longest word omitted by $x_1^* x_2^* \cdots x_k^*$ or $\{x_1, x_2, \ldots, x_k\}^*$, we might consider their state complexity.

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The *state complexity* of a regular language L is the smallest number of states in any DFA that accepts L. It is written $\mathrm{sc}(L)$.

It turns out that the state complexity of $\{x_1, x_2, \ldots, x_k\}^*$ can be exponential in both the length of the longest word and the number of words.

State Complexity

Theorem. Let t be an integer ≥ 2, and define words as follows:

$$y := 01^{t-1}0$$

and

$$x_i := 1^{t-i-1}01^{i+1}$$

for $0 \leq i \leq t-2$. Let $S_t := \{0, x_0, x_1, \ldots, x_{t-2}, y\}$. Then S_t^* has state complexity $3t2^{t-2} + 2^{t-1}$.

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Example. For $t = 6$ the words in S_t are 0 and

$$\begin{aligned} y &= 0111110 \\ x_0 &= 1111101 \\ x_1 &= 1111011 \\ x_2 &= 1110111 \\ x_3 &= 1101111 \\ x_4 &= 1011111 \end{aligned}$$

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Theorem. Let y and x_i be as defined above. Let $L = (0^* x_1^* x_2^* \cdots x_{n-1}^* y^*)^e$ where $e = (t+1)(t-2)/2 + 2t$. Then $\mathrm{sc}(L) \geq 2^{t-2}$.

This example is due to Jui-Yi Kao.

Complexity

Theorem. If S, a finite list of words, is represented by either an NFA or a regular expression, then determining if S^* is co-finite is NP-hard and is in PSPACE.

Theorem. If S is a unary language (possibly infinite) represented by an NFA, then we can decide in polynomial time if S^* is co-finite.

Open Problem

We still do not know the complexity of the following problem:

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Given a finite list of words $S = \{x_1, x_2, \ldots, x_k\}$, determine if S^* is co-finite.

For Further Reading

- J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, 2005.
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