Fife's Theorem for $\frac{7}{3}$-Powers

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More generally, an nth power is a nonempty word of the form $x^n = \overbrace{xx\cdots x}^{n}$.

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For example, `alfalfa` is a $\frac{7}{3}$-power, since it is of length 7 and is 3-periodic.

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An example in Czech is `jejej`.

Thue and overlap-free words

Axel Thue proved that the Thue-Morse word

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The Thue-Morse word can also be viewed in another way: as the fixed point of the Thue-Morse morphism μ sending $0 \rightarrow 01$, $1 \rightarrow 10$.

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Can we somehow characterize all infinite overlap-free binary words?

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Example: the canonical decomposition of 001001101001 is

$$0010 \quad 0110 \quad 1001.$$

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Fife proved that every infinite overlap-free word has a unique description of the form $\mathbf{x}(01)$, $\mathbf{x}(001)$, $\mathbf{x}(10)$, or $\mathbf{x}(110)$, where $\mathbf{x}$ is an infinite word over the alphabet α, β, γ satisfying certain properties.

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These properties amount to specifying a finite automaton accepting the set of valid words.

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- verifying the automaton is complicated
- not clear how to extend this to other kinds of repetitions, such as $\frac{7}{3}$-powers

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Theorem.
Every finite binary overlap-free word w can be written uniquely in the form $x\mu(y)z$, where y is overlap-free, and $x, z \in \{\epsilon, 0, 00, 1, 11\}$.

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Furthermore, if $|w| \geq 7$, then this decomposition is unique.

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Furthermore, the correct decomposition can be deduced by examining the first 5 symbols of $\mathbf{w}$.

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Otherwise, we get

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for $a \in \{0, 1\}$.

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So $\mathbf{h}$ is encoded by the sequence of indices $2313131\cdots = 2(31)^\omega$.

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Yes, using a finite automaton.

The automaton

Let $\mathcal{O}$ denote the set of all infinite overlap-free words.
States of the automaton represent subsets of $\mathcal{O}$, as follows:

$$\begin{aligned}
A &= \mathcal{O} \\
B &= \{\mathbf{x} \in \Sigma^\omega \ : \ 1\mathbf{x} \in \mathcal{O}\} \\
C &= \{\mathbf{x} \in \Sigma^\omega \ : \ 1\mathbf{x} \in \mathcal{O} \text{ and } \mathbf{x} \text{ begins with } 101\} \\
D &= \{\mathbf{x} \in \Sigma^\omega \ : \ 0\mathbf{x} \in \mathcal{O}\} \\
E &= \{\mathbf{x} \in \Sigma^\omega \ : \ 0\mathbf{x} \in \mathcal{O} \text{ and } \mathbf{x} \text{ begins with } 010\} \\
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I &= \{\mathbf{x} \in \Sigma^\omega \ : \ 1\mathbf{x} \in \mathcal{O} \text{ and } \mathbf{x} \text{ begins with } 00\} \\
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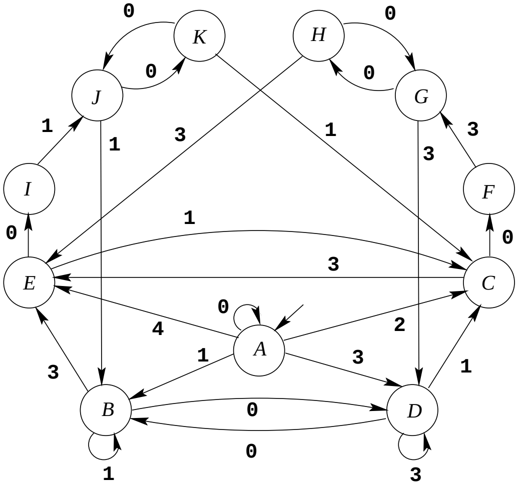
K &= \{\mathbf{x} \in \Sigma^\omega \ : \ 0\mathbf{x} \in \mathcal{O} \text{ and } \mathbf{x} \text{ begins with } 0\}
\end{aligned}$$

We connect states as follows: an arrow from state S to state T is labeled i means

$$\mathbf{w} \in T \iff p_i\, \mu(\mathbf{w}) \in S.$$



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Theorem.
Every infinite binary overlap-free word $\mathbf{x}$ is encoded by an infinite path, starting in state A, through the automaton.

Every infinite path through the automaton not ending in 0^ω codes a unique infinite binary overlap-free word $\mathbf{x}$. If a path $\mathbf{i}$ ends in 0^ω and this suffix corresponds to a cycle on state A or a cycle between states B and D, then $\mathbf{x}$ is coded by either $\mathbf{i}; 0$ or $\mathbf{i}; 1$. If a path $\mathbf{i}$ ends in 0^ω and this suffix corresponds to a cycle between states J and K, then $\mathbf{x}$ is coded by $\mathbf{i}; 0$. If a path $\mathbf{i}$ ends in 0^ω and this suffix corresponds to a cycle between states G and H, then $\mathbf{x}$ is coded by $\mathbf{i}; 1$.

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Currie & Rampersad (2010) showed that $\frac{7}{3}$ is the infimum of all exponents α such that there exists an infinite word avoiding α-powers and containing arbitrarily large squares beginning at every position.

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Partial results in the Ph. D. thesis of Narad Rampersad (2007)

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In this talk: a simpler version, but for infinite words.

Extending Fife to $\frac{7}{3}$-Powers

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Relies on a version of the Restivo-Salemi decomposition that works for $\frac{7}{3}$-powers:

Theorem.
Let $2 < \alpha \leq \frac{7}{3}$. Then every infinite binary α-power-free word $\mathbf{w}$ can be written uniquely in the form

$$\mathbf{w} = x\,\mu(\mathbf{y})$$

where $x \in \{\epsilon, 0, 1, 00, 11\}$ and $\mathbf{y}$ is overlap-free.

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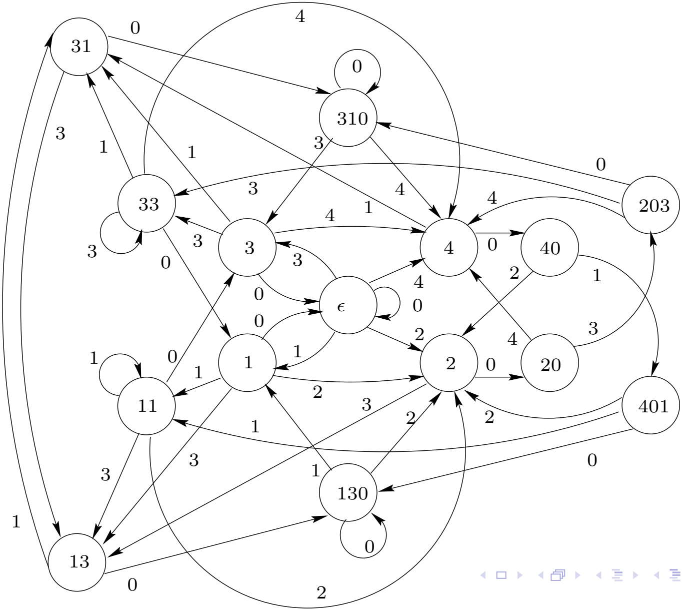
Furthermore, the correct decomposition can be deduced by examining the first 5 symbols of $\mathbf{w}$.

The Fife-like automaton for $\frac{7}{3}$-powers



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Many of these follow from known results on α-power-free words.

Others require some (fairly simple) ad hoc reasoning.

Proof of one assertion

$F_{23} = F_{13}$: in other words, $00\mu(1\mu(\mathbf{w}))$ is $\frac{7}{3}$-power-free iff $0\mu(1\mu(\mathbf{w}))$ is $\frac{7}{3}$-power-free.

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For the other, note that if $0\mu(1\mu(\mathbf{w}))$ is $\frac{7}{3}$-power-free, but $00\mu(1\mu(\mathbf{w}))$ is not, then the $\frac{7}{3}$-power in it must be a prefix.

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However, if $0\mu(1\mu(\mathbf{w}))$ is $\frac{7}{3}$-power-free, then $\mathbf{w}$ must start with 0 (else $0\mu(1\mu(\mathbf{w}))$ would start with 01010).

So $00\mu(1\mu(\mathbf{w}))$ starts with 001001. But this word cannot appear twice, because any letter that precedes it gives a $\frac{7}{3}$-power.

Consequences of the main theorem

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Proof. Examine the possible paths in the automaton.

More consequences of the main theorem

An infinite word $(a_n)_{n\geq 0}$ is k-automatic if there is an automaton with output that, on input n in base k, reaches a state whose associated output is a_n.

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Theorem. An infinite $\frac{7}{3}$-power-free word is 2-automatic if and only if (a) it is encoded by the automaton previously shown and (b) the sequence of symbols coding it is ultimately periodic.

Proof of 2-automatic result

It suffices to look at the 2-decimation of

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But if it is finite then for some $i < j$ we have

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so the x_i are ultimately periodic with period $j - i$.

On the other hand, if the x_i are ultimately periodic then the set of all decimations is finite, since we can specify any decimation by (1) an initial term of length at most 4 (2) whether subsequent terms are complemented and (3) which of a finite set of x_i begins the second term.

For Further Reading

- J. Berstel. A rewriting of Fife's theorem about overlap-free words. In J. Karhumäki, H. Maurer, and G. Rozenberg, editors, *Results and Trends in Theoretical Computer Science*, Vol. 812 of *Lecture Notes in Computer Science*, pp. 19–29. Springer-Verlag, 1994.
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