The Critical Exponent is Computable for Automatic Sequences

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The exponent of the Czech words `jajaj` and `jejej` is $5/2$.

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- as the infinite word generated by an automaton taking the base-k expansion of n as input, and producing the n'th term of the sequence as output

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It can be rational & not be attained: the word $210201210120210201202101210 \cdots$, which counts the run lengths of 1's in $\mathbf{t}$, and is generated by $2 \rightarrow 210$, $1 \rightarrow 20$, and $0 \rightarrow 1$, has critical exponent 2, but it is not attained.

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Further, for words that are fixed points of morphisms, the critical exponent lies in the field extension generated by the eigenvalues of the associated incidence matrix (Krieger, 2006).

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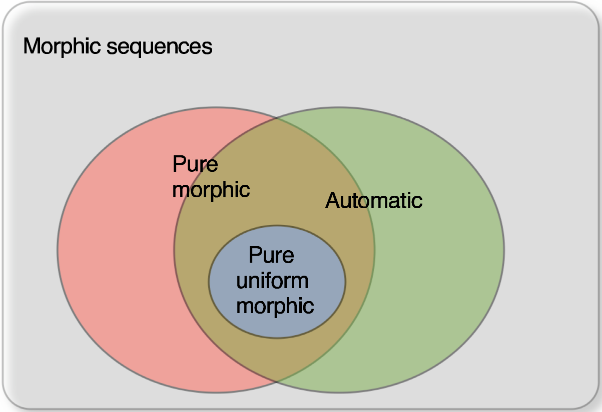
- it is decidable, for an infinite word generated by iterating an arbitrary morphism, if its critical exponent is $< \infty$ (Ehrenfeucht & Rozenberg, 1983)

- in this talk: it is rational and computable for all k-automatic sequences

Morphic, pure morphic, and automatic sequences



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This gives a 1–1 correspondence between $\mathbb{N}$ and elements of $\{\epsilon\} \cup (\Sigma_k - \{0\})\Sigma_k^*$.

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For example, if $m = 23$ and $n = 10$, then $(m)_2 = 10111$ and $(n)_2 = 1010$. We pad the representation of n to get 01010 and then pair up with the digits of m to get

$$(23, 10)_2 = [1, 0][0, 1][1, 0][1, 1][1, 0].$$

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In a similar way, we can encode a set of pairs of integers as a language over $(\Sigma_k^2)^*$.

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A given rational may have one or more representations in L.

The main theorem

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Further, given an automaton for L, we can compute α.

Outline of the proof

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2. Therefore we can find $x \in L$ with $f(x)$ arbitrarily close to α.

3. Using the pumping lemma, we construct $x' \in L$ with $f(x') > \alpha$, a contradiction.

A useful lemma

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where n is the number of states in the minimal DFA accepting L.

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2. The optimal constant for linear recurrence of k-automatic words is either infinite or rational. Furthermore, it is computable.

3. The quantities $\limsup_{n \to \infty} F(n)/n$ and $\liminf_{n \to \infty} F(n)/n$ are computable for automatic sequences, where F denotes subword complexity.

Application to the critical exponent

Given an automaton generating the k-automatic sequence $\mathbf{a} = a_0 a_1 \cdots$, we transform it into an automaton M accepting

$$L = \{(p, q)_k \ : \ \exists \text{ a factor of } \mathbf{a} \text{ of length } q \text{ with period } p\ \}.$$

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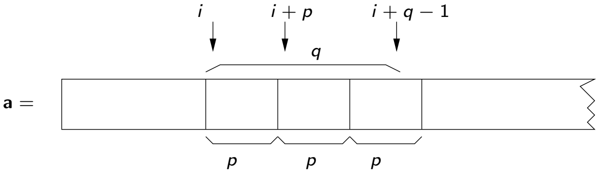

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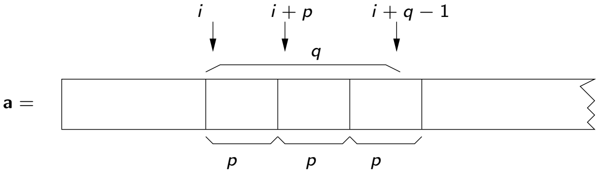

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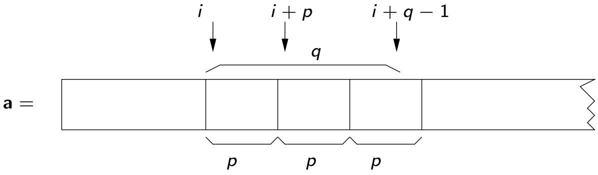

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The critical exponent of $\mathbf{a}$ is then $\sup_{x \in L} f(x)$, which is either rational or infinite.

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Theorem.
If an automatic sequence $\mathbf{a}$ is linearly recurrent, then the optimal constant C is rational and computable.

Sketch of the proof

Define

$L = \{(n, l)_k \;:\;$ (a) there exists $i \geq 0$ s. t. for all $j, 0 \leq j < \ell$ we have $\mathbf{a}[i+j] = \mathbf{a}[i+n+j]$ and
(b) there is no $t, 0 < t < n$ s. t. for all $j, 0 \leq j < \ell$ we have $\mathbf{a}[i+j] = \mathbf{a}[i+t+j]\}$

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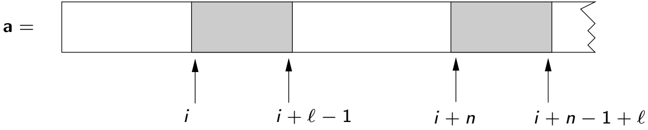

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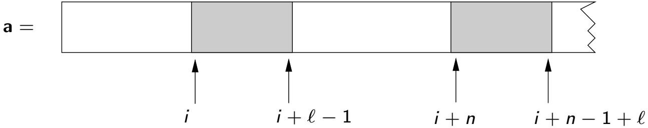

Another way to say this is that L consists of the base-k representation of those pairs of integers (n, ℓ) such that (a) there is some factor of length ℓ for which there is another occurrence at distance n and (b) this occurrence is actually the very next occurrence.

Sketch of the proof

Now from our Theorem we know $\sup\{n/\ell \ : \ (n,\ell)_k \in L\}$ is either infinite or rational. In the latter case this sup is computable, and this gives the optimal constant C for the linear recurrence of $\mathbf{a}$.

Applications to subword complexity

Let $F(n)$ denote the number of distinct factors of length n in a given k-autmatic sequence (aka subword complexity).

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Hence their equality is computable (cf. Goldstein [2011]).

Open problems

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– Is the theorem about $\sup_{x \in L} f(x)$ also true for context-free languages?

For further reading

1. E. Charlier, N. Rampersad, and J. Shallit, Enumeration and decidable properties of automatic sequences. In DLT 2011, pp. 165–179.