Fixed Points of Morphisms

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Finite Fixed Points

Up to now, we have been looking at infinite fixed points of morphisms.

But some morphisms have finite fixed points.

For example, consider the morphism defined by

$$\begin{aligned} h(\mathtt{a}) &= \mathtt{abbc} \\ h(\mathtt{b}) &= \epsilon \\ h(\mathtt{c}) &= \mathtt{d} \\ h(\mathtt{d}) &= \epsilon \end{aligned}$$

Then h has many different finite fixed points, including ϵ and abbcd.
Let's determine all the finite words that are fixed by a morphism.

Finite Fixed Points

Lemma. Let $h : \Sigma^* \to \Sigma^*$ be a morphism. Let $w \in \Sigma^+$ be a finite nonempty word such that w is a subword of $h(w)$. Then there exists a letter $a \in \Sigma$ occurring in w such that a occurs in $h(a)$.

Proof. Let

$$w = c_1 c_2 \cdots c_n,$$

where $c_i \in \Sigma$ for $1 \leq i \leq n$.

For $0 \leq i \leq n$ define

$$s(i) = |h(c_1 c_2 \cdots c_i)|.$$

In particular, $s(0) = 0$.
Let

$$h(w) = d_1 d_2 \cdots d_{s(n)},$$

where $d_i \in \Sigma$ for $1 \leq i \leq s(n)$.

Now

$$h(w) = d_1 d_2 \cdots d_{s(n)},$$

where $d_i \in \Sigma$ for $1 \leq i \leq s(n)$.

Hence

$$h(c_i) = d_{s(i-1)+1} \cdots d_{s(i)}$$

for $1 \leq i \leq n$.

Since w is a subword of $h(w)$, we know there must exist an integer t, $0 \leq t \leq s(n) - n$, such that $w = d_{t+1} \cdots d_{t+n}$.

Hence $c_i = d_{t+i}$ for $1 \leq i \leq n$.
Consider the least index $j \geq 1$ for which $s(j) \geq t + j$.

Such an index must exist, since the inequality holds for $j = n$.

There are now two cases to consider.

Case 1: $j = 1$*:* Then

$$h(c_1) = d_1 d_2 \cdots d_{s(1)}$$

and $s(1) \geq t + 1$. Hence $h(c_1)$ contains $d_{t+1} = c_1$. Let $a = c_1$.

Case 2: $j > 1$: Then by the definition of j we must have $s(j-1) < t + j - 1$.

Hence

$$s(j-1) + 1 < t + j \leq s(j)$$

and since

$$h(c_j) = d_{s(j-1)+1} \cdots d_{s(j)},$$

we know $h(c_j)$ contains $d_{t+j-1} d_{t+j} = c_{j-1} c_j$ as a subword.

Let $a = c_j$.

Finite Fixed Points

Corollary. If $w \in \Sigma^+$ is a nonempty finite word with $h(w) = w$, then there exist words $w_1, w_2, w_3, w_4 \in \Sigma^*$ and a letter $a \in \Sigma$ such that

$$\begin{aligned} w &= w_1 w_2 a w_3 w_4, \\ h(w_1 w_2) &= w_1, \\ h(a) &= w_2 a w_3, \\ h(w_3 w_4) &= w_4. \end{aligned}$$

Proof. If $h(w) = w$, then, using the notation in the proof of the previous lemma, we have $t = 0$ and $s(n) = n$.

Define

$$
\begin{aligned}
w_1 &:= d_1 \cdots d_{s(j-1)}; \\
w_2 &:= d_{s(j-1)+1} \cdots d_{j-1}; \\
a &:= d_j; \\
w_3 &:= d_{j+1} \cdots d_{s(j)}; \\
w_4 &:= d_{s(j)+1} \cdots d_n.
\end{aligned}
$$

Now verify that indeed

$$
\begin{aligned}
h(w_1 w_2) &= w_1, \\
h(a) &= w_2 a w_3, \\
h(w_3 w_4) &= w_4.
\end{aligned}
$$

Mortal Letters

The set of mortal letters associated with a morphism h is denoted by M_h.

The *mortality exponent* of a morphism h is defined to be the least integer $t \geq 0$ such that $h^t(a) = \epsilon$ for all $a \in M_h$.

(If $M_h = \emptyset$, we take $t = 0$.)

We write the mortality exponent as $\exp(h) = t$.

Now define

$$A_h = \{a \in \Sigma \ : \ \exists\, x, y \in M_h^* \text{ such that } h(a) = xay\}$$

and

$$F_h = \{h^t(a) \ : \ a \in A_h \text{ and } t = \exp(h)\}.$$

Note that there is at most one way to write $h(a)$ in the form xay with $x, y \in M_h^*$.

Finite Fixed Points

Theorem. Let $h : \Sigma^* \to \Sigma^*$ be a morphism. Then a finite word $w \in \Sigma^*$ has the property that $w = h(w)$ if and only if $w \in F_h^*$.

Proof. ($\Longleftarrow$): Suppose $w \in F_h^*$. Then we can write $w = w_1 w_2 \cdots w_r$, where each $w_i \in \Sigma^*$, and there exist letters $a_1, a_2, \ldots, a_r \in A_h$ such that $w_i = h^t(a_i)$, with $t = \exp(h)$.

Since $a_i \in A_h$, we know that there exist x_i, y_i with $x_i y_i \in M_h^*$ such that $h(a_i) = x_i a_i y_i$. Since $t = \exp(h)$, we have $h^t(x_i) = h^t(y_i) = \epsilon$. Hence

$$h^{t+1}(a_i) = h^t(x_i)\, h^t(a_i)\, h^t(y_i) = h^t(a_i).$$

Thus $h(w_i) = w_i$ for $1 \leq i \leq r$, and so $h(w) = w$.

Finite Fixed Points

($\Longrightarrow$): We prove the result by contradiction.
Suppose $h(w) = w$, and assume w is the shortest such word with $w \notin F_h^*$.
Clearly $w \neq \epsilon$.

By a previous corollary there exist w_1, w_2, w_3, w_4, a such that $w = w_1 w_2 a w_3 w_4$, $h(w_1 w_2) = w_1$, $h(a) = w_2 a w_3$, and $h(w_3 w_4) = w_4$.

Now a is a subword of w, so $h(a)$ is a subword of $h(w) = w$, and hence by an easy induction, it follows that

$$h^i(a) \text{ is a subword of } w \text{ for all } i \geq 0. \tag{1}$$

Then we must have $w_2 w_3 \in M_h^*$, since otherwise the length of

$$h^i(a) = h^{i-1}(w_2) \cdots h(w_2)\, w_2\, a\, w_3\, h(w_3) \cdots h^{i-1}(w_3)$$

would grow without bound as $i \to \infty$, contradicting (1).
It follows that $h^t(w_2 w_3) = \epsilon$, where $t = \exp(h)$.
Now we have $w_1 = h(w_1 w_2)$, so by applying h^t to both sides, we see

$$\begin{aligned} h^t(w_1) &= h^{t+1}(w_1 w_2) \\ &= h^{t+1}(w_1)\, h^{t+1}(w_2) \\ &= h^{t+1}(w_1). \end{aligned}$$

Hence, defining $y_1 = h^t(w_1)$, we have $h(y_1) = y_1$.

In a similar fashion, if we set $y_2 = h^t(w_4)$, then $h(y_2) = y_2$.

Since $|y_1|, |y_2| < |w|$, it follows by the minimality of w that $y_1, y_2 \in F_h^*$.

Now

$$\begin{aligned} w &= h^t(w) \\ &= h^t(w_1)\, h^t(w_2)\, h^t(a)\, h^t(w_3)\, h^t(w_4) \\ &= y_1\, h^t(a)\, y_2, \end{aligned}$$

and hence $w \in F_h^*$, a contradiction.

How long can a fixed point be?

Suppose h possesses a nonempty finite fixed point w. How long can the shortest w be, as a function of the description of h?

Theorem. If a homomorphism h possesses a nonempty finite fixed point, then there exists such a fixed point w with $|w| \leq m^{n-1}$, where $n = |\Sigma|$ and $m = \max_{a \in \Sigma} |h(a)|$.
Furthermore, this bound is best possible.

Proof. We know a word w is a finite fixed point iff $w \in F_h^*$.

Hence, if there exists a nonempty finite fixed point, the shortest such must lie in F_h.

But

$$F_h = \{h^t(a) \ : \ a \in A_h \text{ and } t = \exp(h)\}.$$

Since $a \in A_h$, we have $h(a) = xay$ with $xy \in M_h^*$.

Hence $a \notin M_h$ and so $\exp(h) \leq |M_h| \leq n - 1$.

If $m = \max_{a \in \Sigma} |h(a)|$, then clearly $|h^i(a)| \leq m^i$ for all $i \geq 0$.

It follows that $|w| = |h^t(a)| \leq m^{n-1}$.

The bound is best possible

To see that the bound m^{n-1} is best possible, consider the homomorphism h defined on $\Sigma = \{a_1, a_2, \ldots, a_n\}$ as follows:

$$\begin{aligned} h(a_1) &= a_1 a_2^{m-1}; \\ h(a_i) &= a_{i+1}^m \text{ for } 2 \le i \le n-1; \\ h(a_n) &= \epsilon. \end{aligned}$$

Then

$$w = a_1 a_2^{m-1} a_3^{m(m-1)} \cdots a_n^{m^{n-2}(m-1)}$$

is a fixed point of h, and

$$|w| = 1 + (m-1) + m(m-1) + \cdots + m^{n-2}(m-1) = m^{n-1}.$$

Example. Consider the homomorphism h defined on $\Sigma = \{a_1, a_2, a_3, a_4, a_5\}$ as follows:

$$\begin{aligned} h(a_1) &= a_1\, a_2 \\ h(a_2) &= a_3\, a_3 \\ h(a_3) &= a_4\, a_4 \\ h(a_4) &= a_5\, a_5 \\ h(a_5) &= \epsilon \end{aligned}$$

Using the notation in the above theorem, $m = 2$ and $n = 5$. The following word w is a finite fixed point such that $|w| \leq 2^4$.

$$\begin{aligned} w &= a_1\, a_2\, a_3^2\, a_4^4\, a_5^8 \quad \text{and} \\ h(w) &= a_1\, a_2\, a_3^2\, a_4^4\, a_5^8. \end{aligned}$$

One-Sided Infinite Fixed Points

Let $\mathbf{w} = c_1 c_2 c_3 \cdots$ be a one-sided right-infinite word over Σ, and let h be a morphism. Our next goal is to characterize those $\mathbf{w}$ for which $h(\mathbf{w}) = \mathbf{w}$.

Let Σ^ω denote the set of all right-infinite words over the alphabet Σ.

If $\mathbf{w} = a_1 a_2 a_3 \cdots$, then $h(\mathbf{w}) = h(a_1) h(a_2) h(a_3) \cdots$.

If $L \subseteq \Sigma^*$ is a language, then we define

$$L^\omega = \{ w_1 w_2 w_3 \cdots \ : \ w_i \in L \setminus \{\epsilon\} \text{ for all } i \geq 1 \}.$$

Perhaps slightly less obviously, we can also define the word $h^\omega(a)$ for a letter a, provided $h(a) = xay$ and $x \in M_h^*$.

In this case, there exists $t \geq 0$ such that $h^t(x) = \epsilon$.

Then we define

$$h^\omega(a) = h^{t-1}(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y) \cdots ,$$

which is infinite if and only if $y \notin M_h^*$.

One-Sided Infinite Fixed Points

Theorem. The right-infinite word $\mathbf{w}$ is a fixed point of h if and only if at least one of the following two conditions holds:

(a) $\mathbf{w} \in F_h^\omega$; or

(b) $\mathbf{w} \in F_h^* h^\omega(a)$ for some $a \in \Sigma$, and there exist $x \in M_h^*$ and $y \notin M_h^*$ such that $h(a) = xay$.

Note that there is at most one way to write $h(a) = xay$ with $x \in M_h^*$ and $y \notin M_h^*$.

Proof. ($\Longleftarrow$): First, suppose condition (a) holds; that is, $\mathbf{w} \in F_h^\omega$.

Then we can write $\mathbf{w} = w_1 w_2 w_3 \cdots$, where each $w_i \in F_h$.

Then by our classification of finite fixed points, we have $h(w_i) = w_i$.

It follows that $h(\mathbf{w}) = \mathbf{w}$.

Second, suppose condition (b) holds, that is, $\mathbf{w} = v\,\mathbf{z}$, where $v \in F_h^*$ and $\mathbf{z} = h^\omega(a)$, where $h(a) = xay$ for some $x \in M_h^*$, $y \notin M_h^*$.

Then from our classification of finite fixed points, we have $h(v) = v$.

Since $x \in M_h^*$, we have $h^t(x) = \epsilon$ where $t = \exp(h)$, and hence

$$\begin{aligned} \mathbf{z} &= h^\omega(a) \\ &= h^{t-1}(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y)\, h^3(y) \cdots . \end{aligned}$$

Since $y \notin M_h^*$, it follows that $|h^i(y)| \geq 1$ for all $i \geq 0$, and hence $\mathbf{z}$ is indeed an infinite word.

We then have

$$\begin{aligned} h(\mathbf{z}) &= h^t(x) \cdots h(x)\, x\, a\, y\, h(y)\, h^2(y)\, h^3(y) \cdots \\ &= \mathbf{z} \end{aligned}$$

and so $h(\mathbf{w}) = h(v\mathbf{z}) = v\mathbf{z} = \mathbf{w}$.

The converse

($\Longrightarrow$): Now suppose $\mathbf{w} = c_1 c_2 c_3 \cdots$ is an infinite word, with $c_i \in \Sigma$ for $i \geq 1$, and $h(\mathbf{w}) = \mathbf{w}$.

As before, we define $s(i) = |h(c_1 c_2 \cdots c_i)|$ for $i \geq 0$.

There are several cases to consider:

Case 1: $s(i) = i$ for infinitely many integers $i \geq 1$.

Case 2: $s(i) = i$ for finitely many $i \geq 1$, and at least one such i.

Case 3: $s(i) \neq i$ for all $i \geq 1$.

Case 1: $s(i) = i$ for infinitely many integers $i \geq 1$.

Suppose $s(i) = i$ for $i = i_0, i_1, i_2, \ldots$.

Clearly we may take $i_0 = 0$.

Then we can write

$$\mathbf{w} = y_1 y_2 y_3 \cdots$$

where $y_j = c_{i_{j-1}+1} \cdots c_{i_j}$ and $h(y_j) = y_j$ for $j \geq 1$. It follows that

$\mathbf{w} \in F_h^\omega$.

Case 2: $s(i) = i$ for finitely many $i \geq 1$, and at least one such i.

Let $i_0 = 0, i_1, \ldots, i_r$, $r \geq 1$, be all the indices i for which $s(i) = i$.

Then we can write

$$\mathbf{w} = y_1 y_2 y_3 \cdots y_r \, \mathbf{x}$$

where $y_j = c_{i_{j-1}+1} \cdots c_{i_j}$, and $h(y_j) = y_j$ for $1 \leq j \leq r$, and $h(\mathbf{x}) = \mathbf{x}$.

Furthermore, if we write $\mathbf{x} = d_1 d_2 d_3 \cdots$ for $d_i \in \Sigma$, $i \geq 1$, then

$$s(i) \neq i \text{ for all } i \geq 1. \qquad (2)$$

If we can show that (2) implies that $\mathbf{x} = h^\omega(a)$, where $h(a) = xay$ for some $x \in M_h^*$, $y \notin M_h^*$, we will be done. This leads to Case 3.

Case 3: $s(i) \neq i$ for all $i \geq 1$.

Suppose there exist i, j with $1 \leq i < j$ and

$$s(i) > i \text{ but } s(j) < j. \tag{3}$$

Among all pairs (i, j) with $1 \leq i < j$ satisfying (3), choose one with $j - i$ minimal.

Suppose there exists an integer k with $i < k < j$.

If $s(k) < k$, then (i, k) has a smaller difference, while if $s(k) > k$, then (k, j) has a smaller difference. It follows that $j = i + 1$.

Then $s(i) > i$, but $s(i+1) < i+1$, a contradiction, since $s(i) \leq s(i+1)$.

It follows that either (a) $s(i) < i$ for all $i \geq 1$, or (b) there exists an integer $r \geq 1$ such that $s(i) < i$ for $1 \leq i < r$ and $s(i) > i$ for all $i \geq r$.

Case 3a: $s(i) < i$ for all $i \geq 1$.

Since this is true for $i = j_0 := 1$, in particular we see that $h(c_1) = \epsilon$.

Now let j_1 be the least index such that

$$h(c_{j_1}) \text{ contains } c_1; \tag{4}$$

such an index must exist since $h(\mathbf{w}) = \mathbf{w}$.

We then have $h(c_2) = h(c_3) = \cdots = h(c_{j_1-1}) = \epsilon$, so the first occurrence of c_{j_1} in $\mathbf{w}$ is at position j_1.

Now inductively assume that we have constructed a strictly increasing sequence $j_0 < j_1 < \cdots < j_t$ such that the first occurrence of c_{j_i} in $\mathbf{w}$ is at position j_i, for $1 \le i \le t$.

Let j_{t+1} be the least index such that $h(c_{j_{t+1}})$ contains c_{j_t}.

Assume $j_t \ge j_{t+1}$. Since $s(i) < i$ for all i, we have $h(c_{j_{t+1}}) = c_k \cdots c_l$ with $l < j_{t+1} \le j_t$. Since $h(c_{j_{t+1}})$ contains c_{j_t}, this implies that c_{j_t} occurs to the left of position j_t, a contradiction. Hence $j_t < j_{t+1}$.

Thus we can construct an infinite strictly increasing sequence $j_0 < j_1 < \cdots$ such that the first occurrence of c_{j_i} in $\mathbf{w}$ is at position j_i.

It follows that the letters $c_{j_0}, c_{j_1}, \ldots$ in Σ are all distinct. But Σ is finite, a contradiction. Hence this case cannot occur.

Case 3b: There exists an integer $r \geq 1$ such that

$$s_{\mathbf{w}}(i) < i \text{ for } 1 \leq i < r \text{ and } s_{\mathbf{w}}(i) > i \text{ for all } i \geq r. \qquad (5)$$

Put $a = c_r$.

Then $h(a) = xay$ for some $x, y \in \Sigma^*$, and $|y| \geq 1$.

Furthermore, the conditions (5) on s imply that we can write $\mathbf{w} = u\,a\,\mathbf{v}$ and $h(\mathbf{w}) = h(u)\,x\,a\,y\,h(\mathbf{v})$ such that $u = h(u)x$.

An easy induction now gives $h^i(\mathbf{w}) =$

$$h^i(u)\,h^{i-1}(x)\cdots h(x)\,x\,a\,y\,h(y)\cdots h^{i-1}(y)\,h^i(\mathbf{v}) \qquad (6)$$

and

$$u = h^i(u)\,h^{i-1}(x)\cdots h(x)\,x \qquad (7)$$

for all $i \geq 0$.

Since $|u| < \infty$, it follows from letting $i \to \infty$ in Eq. (7) that there exists an integer $j \geq 0$ such that $h^j(x) = \epsilon$.

Hence $x \in M_h^*$, and so $h^t(x) = \epsilon$, where $t = \exp(h)$.

Now $u = h(u)x$, so $h^t(u) = h^{t+1}(u)h^t(x) = h^{t+1}(u)$.

Define $u' = h^t(u)$; then $h(u') = u'$.

Hence, putting $j = |u'|$, it follows that $s(j) = j$. Hence $j = 0$ and $u' = \epsilon$.

One-Sided Infinite Fixed Points

Now, to get a contradiction, suppose that $y \in M_h^*$.

Then $h^t(y) = \epsilon$.

Define $z = h^t(a)$.

Then

$$\begin{aligned} h(z) &= h^{t+1}(a) \\ &= h^t(h(a)) \\ &= h^t(xay) \\ &= h^t(x)\, h^t(a)\, h^t(y) \\ &= h^t(a) \\ &= z. \end{aligned}$$

Hence, putting $j = |z|$, we see that $s(j) = j$, a contradiction since $|z| \geq 1$.

Hence $y \notin M_h^*$.

Now, letting $i \to \infty$ in (6), we see that $\mathbf{w} = h^\omega(a)$.

Left-Infinite Fixed Points

There exists a similar characterization for left-infinite fixed points.

Suppose $h(a) = xay$ with $x \notin M_h^*$ and $y \in M_h^*$.

Then there exists $t \geq 0$ such that $h^t(x) = \epsilon$.

We define

$$\overleftarrow{h}^{\omega}(a) = h^2(x)\, h(x)\, x\, a\, y\, h(y) \cdots h^{t-1}(y).$$

The left-infinite word $\mathbf{w}$ is a fixed point of h if and only if at least one of the following two conditions holds:

(a) $\mathbf{w} \in {}^{\omega}F_h$; or

(b) $\mathbf{w} \in \overleftarrow{h}^{\omega}(a)F_h^*$ for some $a \in \Sigma$, and there exist $x \notin M_h^*$ and $y \in M_h^*$ such that $h(a) = xay$.

Two-Sided Infinite Fixed Points

Let $\Sigma^{\mathbb{Z}}$ denote the set of all two-sided infinite words over the alphabet Σ, which are of the form $\cdots c_{-2}c_{-1}c_0.c_1c_2\cdots$.

We use a decimal point to the left of the character c_1, to indicate how the word is indexed.

Let $\mathcal{S}$ denote the shift function, where

$$\mathcal{S}^k(\cdots c_{-2}c_{-1}c_0.c_1c_2c_3\cdots) =$$

$$\cdots c_{k-1}c_k.c_{k+1}c_{k+2}\cdots$$

for all $k \in \mathbb{Z}$.

If $\mathbf{w}, \mathbf{x}$ are 2 two-sided infinite words, and there exists an integer k such that $\mathbf{x} = \mathcal{S}^k(\mathbf{w})$, then we call $\mathbf{w}$ and $\mathbf{x}$ *conjugates*, and we write $\mathbf{w} \sim \mathbf{x}$.

$\sim$ is an equivalence relation.

We extend this notation to sets of infinite words, as follows: if L is a set of two-sided infinite words, then by $\mathbf{w} \sim L$ we mean there exists $\mathbf{x} \in L$ such that $\mathbf{w} \sim \mathbf{x}$.

If $i = |wa|$, $h(a) = wax$, and $w, x \notin M_h^*$, then we define

$$\overleftrightarrow{h^{\omega;i}}(a) := \cdots h^2(w)\, h(w)\, w.a\, x\, h(x)\, h^2(x) \cdots ,$$

a two-sided infinite word.

The factorization of $h(a)$ as wax is *not* necessarily unique, and we use the superscript i to indicate which a is being chosen.

We assume $h : \Sigma^* \rightarrow \Sigma^*$ is a morphism that is extended to the domain $\Sigma^{\mathbb{Z}}$ as follows:

$$h(\cdots c_{-2}c_{-1}c_0.c_1c_2\cdots) =$$
$$\cdots h(c_{-2})h(c_{-1})h(c_0).h(c_1)h(c_2)\cdots .$$

We first consider the equation $h(\mathbf{w}) = \mathbf{w}$ for two-sided infinite words.

Two-Sided Infinite Fixed Points

Proposition. The equation $h(\mathbf{w}) = \mathbf{w}$ has a solution if and only if at least one of the following conditions holds:

(a) $\mathbf{w} \in F_h^{\mathbb{Z}}$; or

(b) $\mathbf{w} \in \overleftarrow{h^{\omega}}(a) F_h^{*} . F_h^{\omega}$ for some $a \in \Sigma$, and there exist $x \notin M_h^{*}$, $y \in M_h^{*}$ such that $h(a) = xay$; or

(c) $\mathbf{w} \in {}^{\omega}F_h . F_h^{*} h^{\omega}(a)$ for some $a \in \Sigma$, and there exist $x \in M_h^{*}$, $y \notin M_h^{*}$ such that $h(a) = xay$; or

(d) $\mathbf{w} \in \overleftarrow{h^{\omega}}(a)\, F_h^{*} . F_h^{*} h^{\omega}(b)$ for some $a, b \in \Sigma$ and there exist $x, z \notin M_h^{*}$, $y, w \in M_h^{*}$, such that $h(a) = xay$ and $h(b) = wbz$.

Two-Sided Infinite Fixed Points

Now we characterize the two-sided infinite fixed points of a morphism in the "unpointed" case.

That is, our goal is to characterize the solutions to $h(\mathbf{w}) \sim \mathbf{w}$.

Let's look at some examples. Consider the morphism h defined by

$$\begin{aligned} \mathtt{a} &\rightarrow \epsilon \\ \mathtt{b} &\rightarrow \mathtt{abc} \\ \mathtt{c} &\rightarrow \mathtt{d} \\ \mathtt{d} &\rightarrow \epsilon \end{aligned}$$

Then

$$\cdots \mathtt{cdabcdabcdab.cdabcdabcdab} \cdots$$

is a solution to $h(\mathbf{w}) \sim \mathbf{w}$.

Next, consider the morphism g defined by

$$\begin{aligned} \mathtt{a} &\longrightarrow \mathtt{ba} \\ \mathtt{b} &\longrightarrow \mathtt{a} \\ \mathtt{c} &\longrightarrow \mathtt{cd} \\ \mathtt{d} &\longrightarrow \epsilon \end{aligned}$$

Then

$$\cdots \mathtt{ababaaba.cdcdcd} \cdots$$

is a solution to $g(\mathbf{w}) \sim \mathbf{w}$.

Next, consider the morphism f defined by

$$\begin{aligned} \mathtt{a} &\longrightarrow \mathtt{ab} \\ \mathtt{b} &\longrightarrow \mathtt{ba} \\ \mathtt{c} &\longrightarrow c \end{aligned}$$

Then

$$\cdots \mathtt{cccc.abbabaab} \cdots$$

is a solution to $f(\mathbf{w}) \sim \mathbf{w}$.

Next, consider the morphism v defined by

$$\begin{aligned} \mathtt{a} &\longrightarrow \mathtt{bac} \\ \mathtt{b} &\longrightarrow \mathtt{a} \\ \mathtt{c} &\longrightarrow \epsilon \\ \mathtt{d} &\longrightarrow \mathtt{d} \\ \mathtt{e} &\longrightarrow \mathtt{gef} \\ \mathtt{f} &\longrightarrow \mathtt{fe} \\ \mathtt{g} &\longrightarrow \epsilon \end{aligned}$$

Then

$$\cdots \mathtt{abacbacabacd.dddgeffefegeffegefgeff} \cdots$$

is a solution to $v(\mathbf{w}) \sim \mathbf{w}$.

Next, consider the morphism u defined by

$$\begin{aligned} \mathtt{a} &\longrightarrow \mathtt{b} \\ \mathtt{b} &\longrightarrow \mathtt{abcc} \\ \mathtt{c} &\longrightarrow \mathtt{ac} \end{aligned}$$

Then

$$\cdots \mathtt{abccba.bccacacbacbac} \cdots$$

is a solution to $u(\mathbf{w}) \sim \mathbf{w}$.

Finally, consider the morphism t defined by

$$\begin{aligned} \mathtt{a} &\longrightarrow \mathtt{bb} \\ \mathtt{b} &\longrightarrow \epsilon \\ \mathtt{c} &\longrightarrow \mathtt{aad} \\ \mathtt{d} &\longrightarrow \mathtt{c} \end{aligned}$$

Then

$$\cdots \mathtt{adbbbbcaadbbbbc.aadbbbbcaadbbbbc} \cdots$$

is a solution to $t(\mathbf{w}) \sim \mathbf{w}$.

Classification of two-sided fixed points

Theorem. Let h be a morphism. Then the two-sided infinite word $\mathbf{w}$ satisfies the relation $h(\mathbf{w}) \sim \mathbf{w}$ if and only if at least one of the following conditions holds:

(a) $\mathbf{w} \sim F_h^{\mathbb{Z}}$; or

(b) $\mathbf{w} \sim \overleftarrow{h}^{\omega}(a) \,.\, F_h^{\omega}$ for some $a \in \Sigma$, and there exist $x \notin M_h^*$ and $y \in M_h^*$ such that $h(a) = xay$; or

(c) $\mathbf{w} \sim {}^{\omega}F_h \,.\, h^{\omega}(a)$ for some $a \in \Sigma$, and there exist $x \in M_h^*$ and $y \notin M_h^*$ such that $h(a) = xay$; or

(d) $\mathbf{w} \sim \overleftarrow{h}^{\omega}(a) \,.\, F_h^* h^{\omega}(b)$ for some $a, b \in \Sigma$ and there exist $x, z \notin M_h^*$, $y, w \in M_h^*$, such that $h(a) = xay$ and $h(b) = wbz$; or

(e) $\mathbf{w} \sim \overleftrightarrow{h}^{\omega;i}(a)$ for some $a \in \Sigma$, and there exist $x, y \notin M_h^*$ such that $h(a) = xay$ with $|xa| = i$; or

(f) $\mathbf{w} = (xy)^{\mathbb{Z}}$ for some $x, y \in \Sigma^+$ such that $h(xy) = yx$.

For further reading

J. Shallit and Ming-wei Wang, On two-sided infinite fixed points of morphisms, *Theoret. Comput. Sci.* **270** (2002), 659–675.