Periodicity and Repetitions in Automatic Sequences

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- if **w** is uniformly recurrent.

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If $\Sigma = \Delta$, we can iterate h, writing h^2 for $h \circ h$, etc.

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$$\mu^\omega(0) = 0110100110010110\cdots,$$

which is **t**, the *Thue-Morse* word.

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- An infinite word is said to be *morphic* if it is the image (under a *coding*) of a pure morphic word.
- An infinite word is said to be *automatic* if it is the image (under a coding) of a word generated by iterating a k-uniform morphism.

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- Cassaigne (1994) gave a general decision procedure for certain kinds of HD0L words.
- Krieger (2007, 2008) gave a procedure to compute the critical exponent for binary uniform morphisms and an "almost algorithm" for arbitrary non-erasing morphisms.

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- Will show how this idea can be applied to other kinds of questions

Automatic sequences

But first, one more observation:

- A sequence $(a_n)_{n\geq 0}$ is k-automatic if there is an automaton with output such that, after feeding in n expressed in base k, you arrive at a state with an output of a_n.

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- Cobham proved: a sequence is k-automatic iff it is k-uniform morphic.

Example: the Thue-Morse sequence

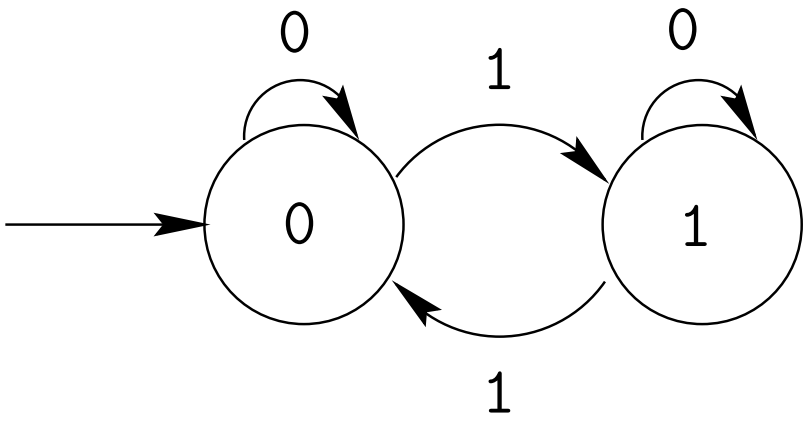

Figure: Automaton generating the Thue-Morse sequence

Testing if an automatic sequence is ultimately periodic

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But both methods are rather complicated.

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Our approach:

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- Interchange accepting and nonaccepting states, obtaining a DFA M_3 such that M_3 accepts (P, N) iff $a_I = a_{I+P}$ for all $I \geq N$.
- Now $\mathbf{a}$ is ultimately periodic iff M_3 accepts some input with $P \geq 1$, which can be checked using depth-first search to see if there is a path from M_3's initial state to a final state.

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Saying that the digits are fed in in parallel means the input alphabet is $\Sigma_k \times \Sigma_k$.

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To handle this, we modify the accepting states, allowing a state to be accepting if we could reach it by following a path labeled with $\overbrace{(0,0)\cdots(0,0)}^{j}$.

Applying the same idea

We can use the same idea to solve other problems, provided we can express our decision problem as a predicate involving quantifiers, addition, and inequalities.

Testing for the presence of overlaps

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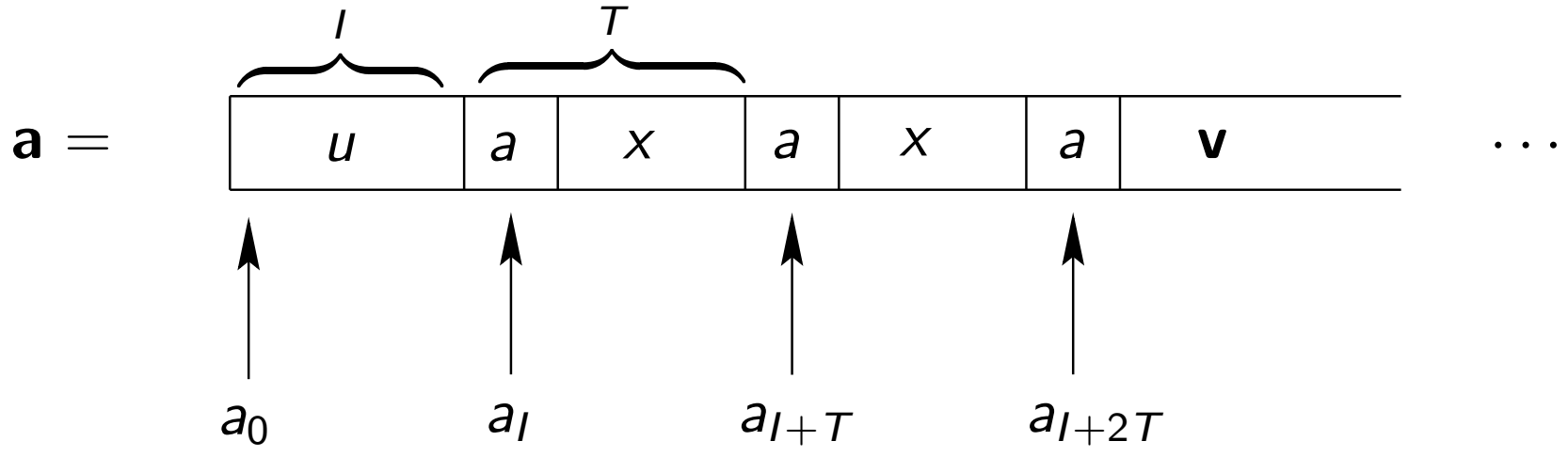

Figure: Hypothesized overlap

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- We then minimized, obtaining a DFA with 2 states accepting only words with $T = 0$. Thus $\mathbf{t}$ is overlap-free.

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Thus whether a k-automatic sequence has a particular fractional power is decidable.

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- Avoids palindromes of length $\geq L$;
- Satisfies the property that if x is a factor, then x^R is not, for all x of length $\geq L$.

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Equivalently, for every $N \geq 0$, $K \geq 1$, there exists $M > N$ such that $a_{N+I} = a_{M+I}$ for $0 \leq I < K$.

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Better results have recently been obtained by Nicolas and Pritykin.

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- If $\mathbf{a}$ is recurrent but not periodic, then the orbit closure is uncountable, but this is not necessarily true if $\mathbf{a}$ is not recurrent.

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For example, the lexicographically least sequence in the orbit closure of $\mathbf{t}$ is the sequence obtained from the complement of $\mathbf{t}$ by dropping the first letter:

$$001011001101001\cdots$$

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Orbit closure

We can define the lexicographically least sequence in the orbit closure as follows:

Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence, and let $\mathbf{b} = (b_n)_{n \geq 0}$ be the lexicographically least sequence in the orbit closure of $\mathbf{a}$. Then $b_I = c$ if and only if there exists $J \geq 0$ such that $a_{J+I} = c$ and $a_L a_{L+1} \cdots a_{L+I} \geq a_J a_{J+1} \cdots a_{J+I}$ for all $L \geq 0$.

Theorem If $\mathbf{a} = (a_n)_{n \geq 0}$ is a k-automatic sequence then so is the lexicographically least sequence in the orbit closure of $\mathbf{a}$.

Continued fractions

Every real number α can be expressed essentially uniquely as a *continued fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}}$$

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which is usually abbreviated

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Here the a_i are positive integers except possibly for a_0, which can be any integer, and the expansion terminates iff α is irrational. The a_i are called *partial quotients*.

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Galois proved that

$$q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$$

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Theorem. Suppose $\alpha = [a_0, a_1, \ldots,]$ is an irrational real number with a k-automatic continued fraction expansion. Then $\limsup_{n\to\infty} q_n/q_{n-1}$ does too.

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The idea of the proof is similar to what we have seen, but there are some small complications arising from the fact that the ordering on continued fraction expansions is not quite the lexicographic order (it reverses alternately at odd and even positions).

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- critical exponent of Sturmian words (Damanik & Lenz; Cao & Wen)

An example

For integers $k \geq 3$ the real number

$$\alpha_k = \sum_{i \geq 0} k^{-2^i} = [0, k-1, k+2, k, k, k-2, k, k+2, k, k-2, k+2, k, \ldots]$$

has a 2-automatic continued fraction expansion, as given below:

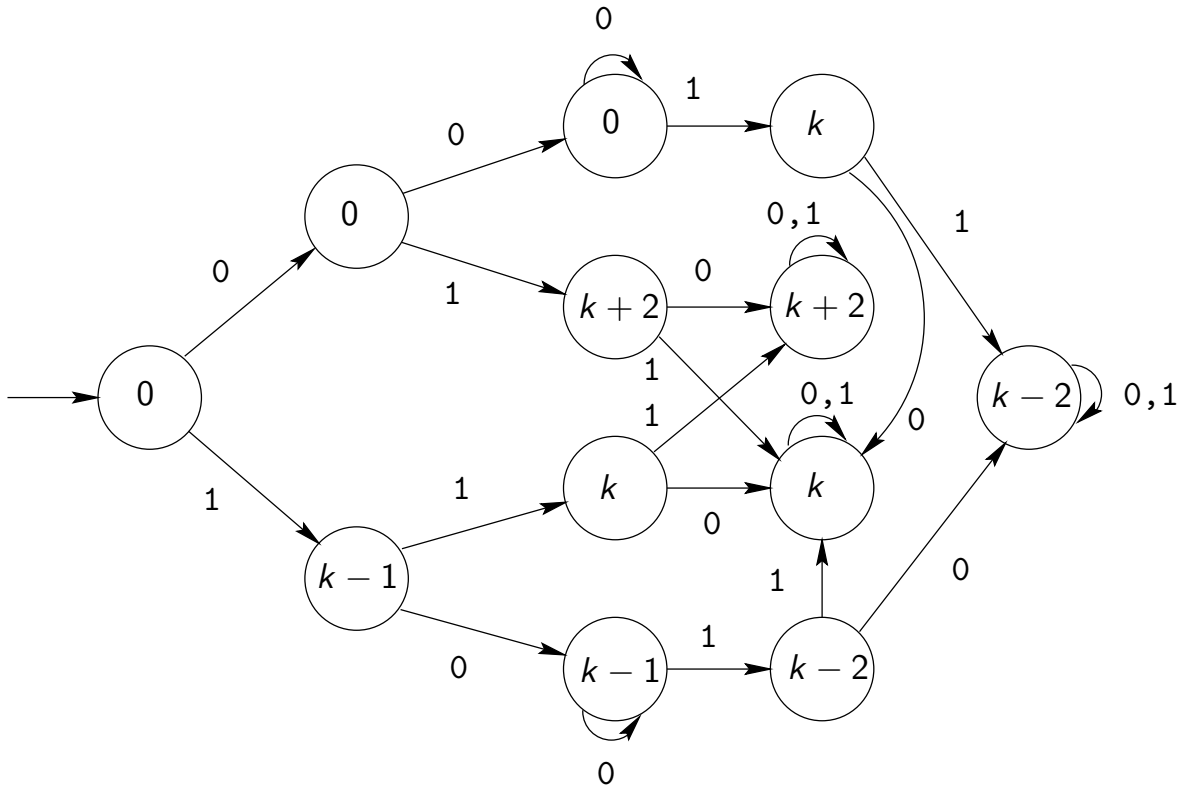

Figure: Automaton generating the continued fraction for α_k

An example

Then
$\zeta = \limsup_{n\geq 0} q_n/q_{n-1} = [k+2, k-2, k, k+2, k, k-2, k, k, \ldots]$
is 2-automatic.

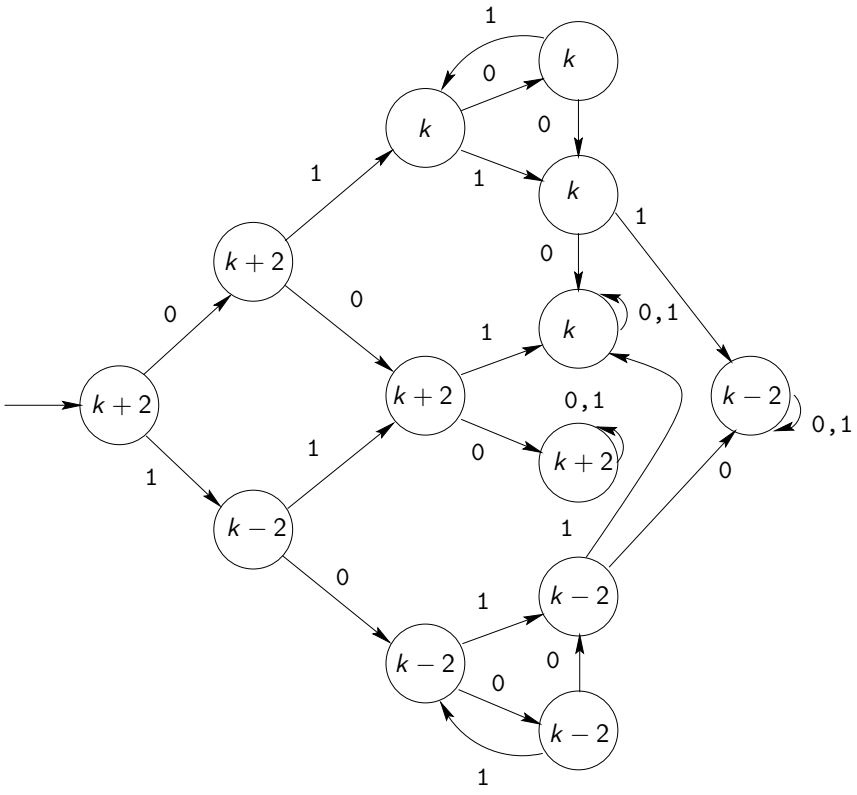

Figure: Automaton generating the continued fraction for ζ

An open problem

Let $\mathbf{r} = (r_n)_{n \geq 0}$ be the Rudin-Shapiro sequence, defined as follows: r_n is the parity of the number of occurrences of 11 in the binary expansion of n.

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Then the lexicographically least sequence in the orbit closure of $\mathbf{r}$ seems to be the sequence obtained by concatenating 0 on the front of $\mathbf{r}$.

But we have not succeeded in proving this rigorously.

Another open problem

Extend all these ideas to arbitrary morphic sequences, not just uniform ones.

For Further Reading

- F. Nicolas and Yu. Pritykin, On uniformly recurrent morphic sequences, *Int. J. Found. Comput. Sci.*, to appear.
- J.-P. Allouche, N. Rampersad, and J. Shallit, Periodicity, repetitions, and orbits of an automatic sequence. *Theoret. Comput. Sci.*, to appear.