Decidability in Automatic Sequences

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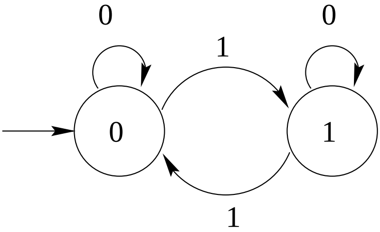
What is an automatic sequence?

- An infinite sequence

$$\mathbf{a} = a_0 a_1 a_2 \cdots$$

  over a finite alphabet of letters, generated by a finite-state machine (automaton)
- The automaton, given n as input, computes a_n as follows:
  - n is represented in some fixed integer base $k \geq 2$
  - The automaton moves from state to state according to this input
  - Each state has an output letter associated with it
  - The output on input n is the output associated with the last state reached

The canonical example: the Thue-Morse automaton



This automaton generates the Thue-Morse sequence

$$\mathbf{t} = (t_n)_{n\geq 0} = 0110100110010110\cdots .$$

Why automatic sequences?

- A nontrivial class of self-similar sequences
- Many “naturally-occurring” sequences are automatic
- Halfway between periodic and chaotic
- Provide canonical examples for various kinds of avoidance problems

Historically interesting properties of $\mathbf{t}$

1. $\mathbf{t}$ is not ultimately periodic.
2. $\mathbf{t}$ contains no factor that is an *overlap*, that is, a word of the form $axaxa$, where a is a single letter and x is an arbitrary finite word. (Example in English: `alfalfa`.)
3. $\mathbf{t}$ contains infinitely many distinct square factors xx, but for each such factor we have $|x| = 2^n$ or $3 \cdot 2^n$, for $n \geq 0$.
4. $\mathbf{t}$ has infinitely many distinct palindromic factors (A *palindrome* is a word equal to its reverse, like `radar`.)
5. The number $p(n)$ of distinct palindromic factors of length n in $\mathbf{t}$ is given by

$$p(n) = \begin{cases} 0, & \text{if } n \text{ odd and } n \geq 5; \\ 1, & \text{if } n = 0; \\ 2, & \text{if } 1 \leq n \leq 4, \text{ or } n \text{ even and } 3 \cdot 4^k + 2 \leq n \leq 4^{k+1}; \\ 4, & \text{if } n \text{ even and } 4^k + 2 \leq n \leq 3 \cdot 4^k. \end{cases}$$

Historically interesting properties of $\mathbf{t}$

6. $\mathbf{t}$ is *mirror-invariant*: if x is a finite factor of $\mathbf{t}$, then so is its reverse x^R.
7. $\mathbf{t}$ is *recurrent*, that is, every factor that occurs, occurs infinitely often.
8. $\mathbf{t}$ is *uniformly recurrent*, that is, for all factors x occurring in $\mathbf{t}$, there is a constant $c(x)$ such that two consecutive occurrences of x are separated by at most $c(x)$ symbols.
9. $\mathbf{t}$ is *linearly recurrent*, that is, it is uniformly recurrent and furthermore there is a constant C such that $c(x) \leq C|x|$ for all factors x. In fact, the optimal bound is given by $c(1) = 3$, $c(2) = 8$, and $c(n) = 9 \cdot 2^e$ for $n \geq 3$, where $e = \lfloor \log_2(n-2) \rfloor$.

Historically interesting properties of $\mathbf{t}$

10. The lexicographically least sequence in the shift orbit closure of $\mathbf{t}$ is $\overline{t_1}\ \overline{t_2}\ \overline{t_3}\ \cdots$, which is also 2-automatic.
11. The *subword complexity* $\rho(n)$ of $\mathbf{t}$, which is the function counting the number of distinct factors of $\mathbf{t}$, is given by

$$\rho(n) = \begin{cases} 2^n, & \text{if } 0 \leq n \leq 2; \\ 2n + 2^{t+2} - 2, & \text{if } 3 \cdot 2^t \leq n \leq 2^{t+2} + 1; \\ 4n - 2^t - 4, & \text{if } 2^t + 1 \leq n \leq 3 \cdot 2^{t-1}; \end{cases} .$$

12. $\mathbf{t}$ has an unbordered factor of length n if $n \not\equiv 1 \pmod 6$ (Here by an *unbordered* word y we mean one with no expression in the form $y = uvu$ for words u, v with u nonempty.)

The claim

Claim. All of these properties can be verified (and in some case, even obtained) purely mechanically, by a machine computation.

To see how, we need to digress into...

Logic

- Let $\mathrm{Th}(\mathbb{N}, +, 0, 1, <)$ denote the set of all true first-order sentences in the logical theory of the natural numbers with addition.
- Example: in this theory we can express the so-called "Chicken McNuggets" theorem that 43 is the largest integer that cannot be represented as a non-negative integer linear combination of $6, 9$, and 20, as follows:

$$(\forall n > 43\ \exists x, y, z \geq 0 \text{ such that } n = 6x + 9y + 20z) \wedge \neg(\exists x, y, z \geq 0 \text{ such that } 43 = 6x + 9y + 20z). \quad (1)$$

  Here, of course, "$6x$" is shorthand for the expression "$x + x + x + x + x + x$", and similarly for $9y$ and $20z$.
- Presburger proved that $\mathrm{Th}(\mathbb{N}, +, 0, 1, <)$ is *decidable*: that is, there exists an algorithm that, given a sentence in the theory, will decide its truth.

Decidability of Presburger arithmetic: proof sketch

- represent integers in an integer base $k \geq 2$ using the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$.
- represent n-tuples of integers as words over the alphabet Σ_k^n, padding with leading zeroes, if necessary.
- For example, the pair $(21, 7)$ can be represented in base 2 by the word

$$[1, 0][0, 0][1, 1][0, 1][1, 1].$$

Decidability of Presburger arithmetic

- Then the relation $x + y = z$ can be checked by a simple 2-state automaton depicted below, where transitions not depicted lead to a nonaccepting "dead state".

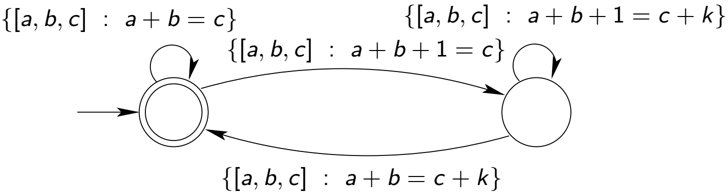


Figure: Checking addition in base k

Decidability of Presburger arithmetic: proof sketch

- Relations like $x = y$ and $x < y$ can be checked similarly.
- Given a formula with free variables $x_1, x_2, \ldots, x_n$, we construct an automaton accepting the base-k expansion of those n-tuples $(x_1, \ldots, x_n)$ for which the proposition holds.
- If a formula is of the form $\exists x_1, x_2, \ldots x_n \; p(x_1, \ldots, x_n)$, then we use nondeterminism to "guess" the x_i and check them.
- If the formula is of the form $\forall p$, we use the equivalence $\forall p \equiv \neg \exists \neg p$; this may require using the subset construction to convert an NFA to a DFA and then flipping the "finality" of states.
- Finally, the truth of a formula can be checked by using the usual depth-first search techniques to see if any final state is reachable from the start state.

Even more

- If we add the function $V_k : \mathbb{N} \to \mathbb{N}$ to our logical theory, where $V_k(x) = k^n$, and k^n is the largest power of k dividing x, it is still decidable by a similar automaton-based technique.
- By doing so, we gain the capability of deciding many questions about automatic sequences.

Theorem

There is an algorithm that, given a predicate phrased using only the universal and existential quantifiers, indexing into a given automatic sequence $\mathbf{a}$*, addition, subtraction, logical operations, and comparisons, will decide the truth of that proposition.*

We call such a predicate an *automatic predicate*.

The bad news

- The worst-case running time of our algorithm is bounded above by

$$2^{2^{\cdot^{\cdot^{\cdot^{2^{p(N)}}}}}},$$

where the number of 2's in the exponent is equal to the number of quantifiers, p is a polynomial, and N is the number of states needed to describe the underlying automatic sequence.

The good news

- Nevertheless, an implementation often succeeds in veryifying statements in the theory
- More information in the talk of Daniel Goč

Deciding periodicity

- An infinite word **a** is *periodic* if it is of the form $x^\omega = xxx\cdots$ for a finite nonempty word x.
- It is *ultimately periodic* if it is of the form yx^ω for a (possibly empty) finite word y.
- Honkala (1986) proved that ultimate periodicity is decidable for automatic sequences.
- Using our approach: it suffices to express ultimately periodicity as an automatic predicate:

$$\exists p \geq 1, N \geq 0 \; \forall i \geq N \; \mathbf{a}[i] = \mathbf{a}[i+p].$$

- When we run this on the Thue-Morse sequence, we discover (as expected) that **t** is not ultimately periodic.

Repetitions

- Thue (1912) proved that $\mathbf{t}$ contains no overlaps; that is, $\mathbf{t}$ is overlap-free.
- Using our technique, we can express the property of having an overlap *axaxa* beginning at position N with $|ax| = p$, as follows: $\mathbf{a}[N..N+p] = \mathbf{a}[N+p..N+2p]$.
- So the corresponding automatic predicate for $\mathbf{t}$ is

$$\exists p \geq 1, N \geq 0 \ \mathbf{t}[N..N+p] = \mathbf{t}[N+p..N+2p],$$

  or, in other words,

$$\exists p \geq 1, N \geq 0 \ \forall i, 0 \leq i \leq p \ \mathbf{t}[N+i] = \mathbf{t}[N+p+i].$$

- We programmed up our decision procedure and verified that indeed $\mathbf{t}$ is overlap-free.

Critical exponent

- We can define more general repetitions as follows: a word x is an α-power for $\alpha \geq 1$ if we can write $x = y^e y'$ where $e = \lfloor \alpha \rfloor$ and y' is a prefix of y and $|x| = \alpha |y|$.
- For example, `abracadabra` is an $\frac{11}{7}$-power.
- The techniques above suffice to check if a k-automatic sequence has α-powers, using the following predicate:

$$\exists N \geq 0, p, q \geq 1 \ \mathbf{a}[N..N+p-q-1] = \mathbf{a}[N+q..N+p-1] \text{ and } p = \alpha q.$$

- However, this observation alone does not suffice to compute the so-called *critical exponent* of **a**, which is the supremum over all rational α such that **a** has α-power factors.
- It turns out that the critical exponent is also computable for automatic sequences.

Mirror invariance

We can express the property that **a** is mirror-invariant as follows:

$$\forall N \geq 0, \ell \geq 1 \ \exists N' \geq 0 \ \mathbf{a}[N..N+\ell-1] = \mathbf{a}[N'..N'+\ell-1]^R,$$

which is the same as

$$\forall N \geq 0, \ell \geq 1 \ \exists N' \geq 0 \ \forall i, \ 0 \leq i < \ell \ \mathbf{a}[N+i] = \mathbf{a}[N'+\ell-i-1],$$

which can be easily checked by our method.

Recurrence

- We can express the property that **a** is recurrent by saying that for each factor, and each integer M there exists a copy of that factor occurring at a position after M in **a**.
- This corresponds to the following predicate:

$$\forall N, M \geq 0, \ell \geq 1\, \exists M' \geq M \quad \mathbf{a}[N..N{+}\ell{-}1] = \mathbf{a}[M'..M'{+}\ell{-}1].$$

- An easy argument shows that an infinite word **a** is recurrent if and only if each finite factor occurs at least twice. This means that the following simpler predicate suffices:

$$\forall N \geq 0, \ell \geq 1\, \exists M \neq N \quad \mathbf{a}[N..N + \ell - 1] = \mathbf{a}[M..M + \ell - 1].$$

Uniform recurrence

- For uniform recurrence, we need to express the fact that two consecutive occurrences of each factor are separated by no more than C positions.
- Since there are only finitely many factors of each length, we can take C to be the maximum of the constants corresponding to each factor of that length.
- Thus uniform recurrence corresponds to the following predicate:

$$\forall \ell \geq 1 \ \exists C \geq 1 \ \forall N \geq 0 \ \exists M \text{ with } N < M \leq N + C$$
$$\mathbf{a}[N..N+\ell-1] = \mathbf{a}[M..M+\ell-1].$$

Orbit closure

- The *shift orbit* of a sequence $\mathbf{a} = a_0 a_1 a_2 \cdots$ is the set of all sequences under the shift, that is, the set

$$\mathcal{S} = \{a_i a_{i+1} a_{i+2} \cdots \; : \; i \geq 0\}.$$

- The *orbit closure* is the topological closure $\overline{\mathcal{S}}$ under the usual topology.
- In other words, a sequence $\mathbf{b} = b_0 b_1 b_2 \cdots$ is in $\overline{\mathcal{S}}$ if and only if, for each $j \geq 0$, the prefix $b_0 \cdots b_j$ is a factor of $\mathbf{a}$.
- Most sequences in the orbit closure of a k-automatic sequence are not automatic themselves.
- However, we can use our method to show that two distinguished sequences, the lexicographically least and lexicographically greatest sequences in the orbit closure, are indeed k-automatic.
- More in the talk of Narad Rampersad.

Unbordered factors

- A word is *bordered* if it can be expressed as uvu for words u, v with u nonempty, and otherwise it is unbordered.
- Currie and Saari proved that $\mathbf{t}$ has an unbordered factor of length n if $n \not\equiv 1 \pmod 6$.
- However, these are not the only lengths with an unbordered factor; for example,

$$0011010010110100110010110100101$$

  is an unbordered factor of length 31.
- We can express the property that $\mathbf{t}$ has an unbordered factor of length ℓ as follows:

$$\exists N \geq 0 \; \forall j, 1 \leq j \leq \ell/2 \; \mathbf{t}[N..N+j-1] \neq \mathbf{t}[N+\ell-j..N+\ell-1].$$

- Using our technique, we were able to prove

Theorem

There is an unbordered factor of length ℓ in $\mathbf{t}$ if and only iff $(\ell)_2 \notin 1(01^*0)^*10^*1$.

Enumeration

- In many cases we can count the number $T(n)$ of length-n factors of an automatic sequence having a particular property P.
- Here by "count" we mean, give an algorithm A to compute $T(n)$ efficiently, that is, in time bounded by a polynomial in $\log n$.
- Although *finding* the algorithm A may not be particularly efficient, once we have it, we can compute $T(n)$ quickly.

Subword complexity

- Subword complexity counts the number of distinct length-n factors of a sequence.
- To count these factors in an automatic sequence, we create a DFA M accepting the language

$$\begin{aligned} & \{(n, \ell)_k \ : \ \mathbf{a}[n..n+\ell-1] \text{ is the first} \\ & \text{occurrence of the given factor}\} \\ = \ & \{(n, \ell)_k \ : \ \forall n' < n \ \mathbf{a}[n..n+\ell-1] \neq \mathbf{a}[n'..n'+\ell-1]\}. \end{aligned}$$

- the number of n corresponding to a given ℓ is just the number of distinct subwords of length ℓ
- this number can be expressed as the product

$$vM_{a_1} \cdots M_{a_i} w$$

for suitable vectors v, w and matrices $M_0, \ldots, M_{k-1}$, where $a_1 \cdots a_i$ is the base-k representation of ℓ, thus giving an efficient algorithm to compute it.

Enumeration

In a similar way, we can handle

- palindrome complexity (the number of distinct length-n palindromic factors)
- the number of words whose reversals are also factors;
- the number of squares of a given length;
- the number of unbordered factors

and so forth.

What other properties of automatic sequences are decidable?

- A difficult candidate: abelian properties
- We say that a nonempty word x is an *abelian square* if it of the form ww' with $|w| = |w'|$ and w' a permutation of w. (An example in English is the word `reappear`.)
- Luke Schaeffer shows that the predicate for abelian squarefreeness is indeed inexpressible in $\mathrm{Th}(\mathbb{N}, +, 0, 1, <, V_k)$
- Nevertheless, some abelian properties are decidable by other means — see the talk of James Currie

The other talks

- Daniel Goč: Automatic theorem-proving in automatic sequences
- Luke Schaeffer: Abelian powers in automatic sequences are not always automatic
- Narad Rampersad: Extremal words in the shift orbit closure of a morphic sequence
- James Currie: Abelian powers and patterns in words: problems and perspectives