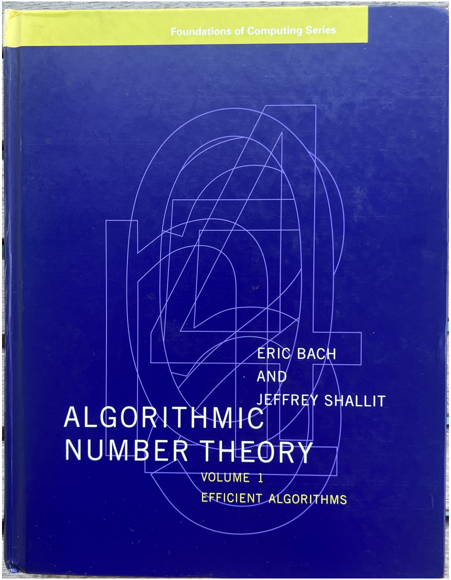
Eric Bach, Berkeley, 1981



Algorithmic Number Theory, 1996

Using Automata to Prove Theorems in Additive Number Theory

Jeffrey Shallit

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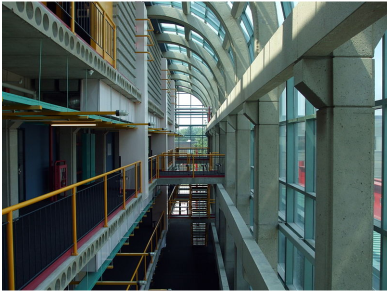

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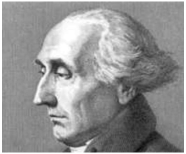

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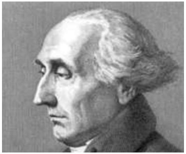

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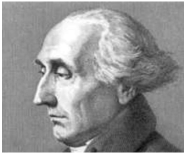

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(Conjectured by Bachet in 1621.)

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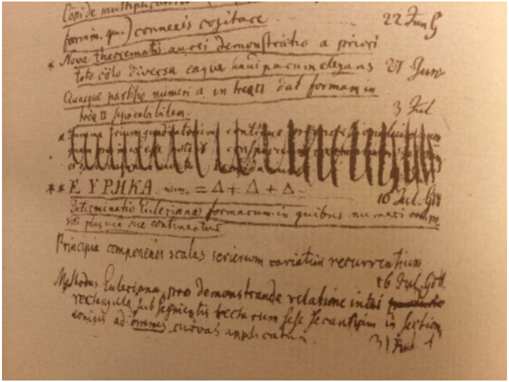

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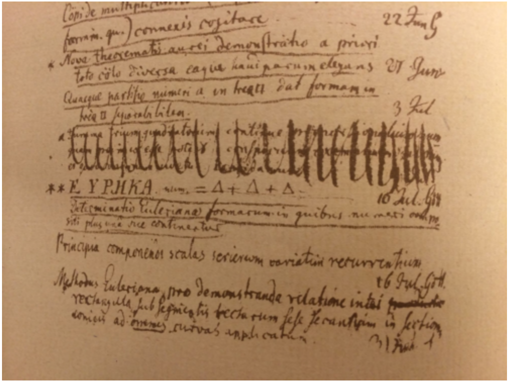

i.e., **The triangular numbers form an additive basis of order** 3

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– "and so forth".

9. Omnis integer numerus vel eſt cubus; vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compoſitus: eſt etiam quadrato-quadratus; vel e duobus, tribus, &c. uſque ad novemdecim compoſitus, & ſic deinceps: conſimilia etiam affirmari poſſunt (exceptis excipiendis) de eodem numero quantitatum earundem dimenſionum.

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We know that $4 \leq G(3) \leq 7$, but the true value is still unknown.

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How about numbers with palindromic base-b expansions?

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- They have density $\Theta(N^{1/2})$.

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Javier Cilleruelo, Florian Luca, and Lewis Baxter (2018) showed that for all bases $b \geq 5$, every natural number is the sum of three base-b palindromes.
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For example,

$$\begin{aligned} 10011938 &= 5127737 + 4851753 + 32447 + 1 \\ &= [10011100011111000111001]_2 \\ &+ [10010100000100000101001]_2 \\ &+ [111111010111111]_2 + [1]_2 \end{aligned}$$

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4 is optimal: 10011938 is not the sum of 2 binary palindromes.

Previous proofs were complicated (1)

Excerpt from Banks (2015):

2.4. **Inductive passage from $\mathbb{N}_{\ell,k}(5^+;c_1)$ to $\mathbb{N}_{\ell-1,k+1}(5^+;c_2)$.**

Lemma 2.4. *Let $\ell, k \in \mathbb{N}$, $\ell \geqslant k+6$, and $c_\ell \in \mathcal{D}$ be given. Given $n \in \mathbb{N}_{\ell,k}(5^+;c_1)$, one can find digits $a_1, \ldots, a_{18}, b_1, \ldots, b_{18} \in \mathcal{D}\backslash\{0\}$ and $c_2 \in \mathcal{D}$ such that the number*

$$n - \sum_{j=1}^{18} q_{\ell-1,k}(a_j, b_j)$$

lies in the set $\mathbb{N}_{\ell-1,k+1}(5^+;c_2)$.

Proof. Fix $n \in \mathbb{N}_{\ell,k}(5^+;c_1)$, and let $\{\delta_j\}_{j=0}^{\ell-1}$ be defined as in (1.1) (with $L := \ell$). Let m be the three-digit integer formed by the first three digits of n; that is,

$$m := 100\delta_{\ell-1} + 10\delta_{\ell-2} + \delta_{\ell-3}.$$

Clearly, m is an integer in the range $500 \leqslant m \leqslant 999$, and we have

$$n = \sum_{j=k}^{\ell-1} 10^j \delta_j = 10^{\ell-3} m + \sum_{j=k}^{\ell-4} 10^j \delta_j. \tag{2.4}$$

Let us denote

$$\mathcal{S} := \{19, 29, 39, 49, 59\}.$$

In view of the fact that

$$9\mathcal{S} := \underbrace{\mathcal{S} + \cdots + \mathcal{S}}_{\text{nine copies}} = \{171, 181, 191, \ldots, 531\},$$

it is possible to find an element $h \in 9\mathcal{S}$ for which $m - 80 < 2h \leqslant m - 60$. With h fixed, let $s_1, \ldots, s_9$ be elements of $\mathcal{S}$ such that

$$s_1 + \cdots + s_9 = h.$$

Previous proofs were complicated (2)

Excerpt from Cilleruelo et al. (2018)

II.2 $\boldsymbol{c_m = 0}$. We distinguish the following cases:

II.2.i) $\boldsymbol{y_m \neq 0}$.

δ_m	δ_{m-1}
0	0
$*$	y_m
$*$	$*$

$\longrightarrow$

δ_m	δ_{m-1}
1	1
$*$	$y_m - 1$
$*$	$*$

II.2.ii) $\boldsymbol{y_m = 0}$.

II.2.ii.a) $\boldsymbol{y_{m-1} \neq 0}$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
y_{m-1}	0	y_{m-1}
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
1	1	$*$
$y_{m-1} - 1$	$g - 2$	$y_{m-1} - 1$
$*$	$z_{m-1} + 1$	$z_{m-1} + 1$

The above step is justified for $z_{m-1} \neq g - 1$. But if $z_{m-1} = g - 1$, then $c_{m-1} \geq (y_{m-1} + z_{m-1})/g \geq 1$, so $c_m = (z_{m-1} + c_{m-1})/g = (g - 1 + 1)/g = 1$, a contradiction.

II.2.ii.b) $\boldsymbol{y_{m-1} = 0,\ z_{m-1} \neq 0}$.

δ_m	δ_{m-1}	δ_{m-2}
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$*$	$z_{m-1} - 1$	$z_{m-1} - 1$

II.2.ii.c) $\boldsymbol{y_{m-1} = 0,\ z_{m-1} = 0}$.

If also $c_{m-1} = 0$, then $\delta_{m-1} = 0$, which is not allowed. Thus, $c_{m-1} = 1$.

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- Idea: could we automate such proofs?

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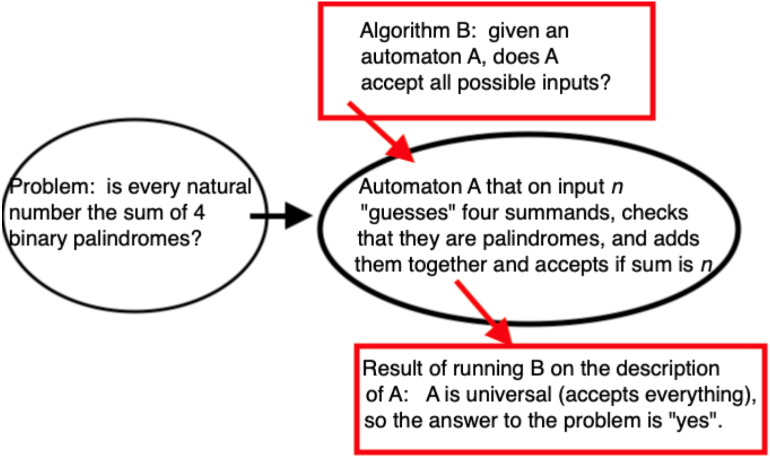
- Then use a decision procedure to establish properties about the language of representations accepted by this machine (e.g., universality – does it accept every possible input?)
- We build the machine, but never run it! What we run is an algorithm that decides a property of the machine.

Our proof strategy



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- The set of all accepted strings is called the *language recognized* by the automaton

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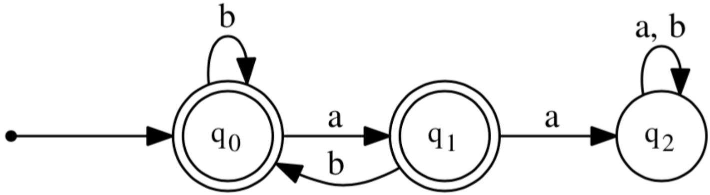
- The transition function that specifies, for each state and each input symbol, which state to enter

Example of an automaton

A double circle represents an accepting state.



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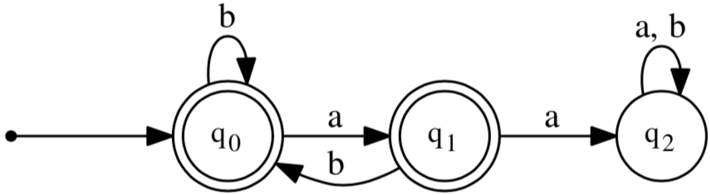

What is the language accepted by this automaton?

It is the set of all strings having no two consecutive a's.

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- In some cases, we can also decide universality: the property of accepting all strings.

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What to do?

A trick

There is a trick that allows us to use finite automata.

The basic idea is the following: there is an 8-state finite automaton that takes four inputs in parallel—numbers represented by strings x, y, z, w—and accepts if and only if $x = ta$ and $y = ub$ and $[tat^R]_2 + [ubu^r] = [zw^R]_2$. Here x and y must start with 1 and so must z. What we get is a description of those n bit numbers that are the sum of two $(n-1)$-bit palindromes.

So we can implicitly represent palindromes by xx^R and xax^R and add them implicitly.

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- We then prove that all inputs accepted by B are accepted by A
- Thus every odd integer ≥ 256 is the sum of three binary palindromes.
- For even integers, we just include 1 as one of the summands.
- The numbers < 256 are easily checked by brute force.
- And so we've proved: every natural number is the sum of four binary palindromes.

Other results

Theorem

Every natural number $N > 256$ *is the sum of at most three base-3 palindromes.*

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This completes the classification for base-b palindromes for all $b \geq 2$.

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- Examples are $36 = [100100]_2$ and $3 = [11]_2$.
- The binary squares form sequence [A020330](https://oeis.org/A020330) in the OEIS

$$3, 10, 15, 36, 45, 54, 63, 136, 153, 170, 187, 204, 221, \ldots$$

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We also have the following result

Theorem

Every natural number is the sum of at most two binary squares and at most two powers of 2.

Generalizing: Waring's theorem for binary k'th powers

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- 1 is not a binary k'th power, so it has to be "every sufficiently large natural number" and not "every natural number".
- The gcd g of the binary k'th powers need not be 1, so it actually has to be "every sufficiently large multiple of g".

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Example:
The binary 6'th powers are

$$63, 2730, 4095, 149796, 187245, 224694, 262143, 8947848, 10066329, \ldots$$

with gcd equal to $\gcd(6, 63) = 3$.

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Obtained with Daniel Kane and Carlo Sanna.

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  (a) the coefficients of $c_k(i)$, $n \le i < n+k$, could be much too large;
  (b) the coefficients could be too small or negative;
  (c) the coefficients might not be integers.

All of these problems can be handled with some work.

Other results

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The proof uses, in part, a decidable extension of Presburger arithmetic.

Congratulations, Eric!

Eric Bach, Gunsight Pass Trail, Glacier National Park, 1987