

New Results on Avoidability in Words

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Squares

- This talk is about words (strings of symbols) over a finite alphabet.
- A nonempty word is called a **square** if it is of the form xx , where x is a word.
- For example, here are some squares in English:
 - atlatl
 - murmur
 - tartar
 - beriberi
 - hotshots
- A word is **squarefree** if it contains no square subwords. (A subword is a block of contiguous symbols inside another word.)
- It is easy to see that every word of length ≥ 4 over the alphabet $\Sigma = \{0, 1\}$ contains a square.

Squarefree Words

- Are there arbitrarily large squarefree words over an alphabet of size 3?
- The Norwegian mathematician Axel Thue proved in 1906 that there are arbitrarily large square-free words (and hence infinite squarefree words) over an alphabet of size 3.

- One such word begins

210201210120210201202101210201210120 . . .

- This word is the **fixed point** of the **morphism** g , which sends $2 \rightarrow 210$, $1 \rightarrow 20$, and $0 \rightarrow 1$.
- His construction was rediscovered many times, for example, by Marston Morse in 1921 and by the Dutch chess master Max Euwe in 1929.

Cubes and Overlaps

- A nonempty word is called a **cube** if it is of the form xxx , where x is a word.
- The English sort-of-word shshsh is a cube, as is the Finnish word kokoko.

- A word is an **overlap** if it is of the form $axaxa$, where a is a single letter and x is a (possibly empty) word.
- The English words
 - alfalfa
 - entente
 - kinnikinnik

are overlaps. You can think of an overlap as a “ $2 + \epsilon$ ” or just “ 2^+ ” power, since it is just slightly larger than a square.

Overlap-free Words

- Thue also proved that there exists an infinite word over a 2-letter alphabet that avoids overlaps (and hence cubes).

- His example begins

01101001100101101001011001101001100...

and is now known as the Thue-Morse word.

- It is the fixed point of the morphism μ , which sends $0 \rightarrow 01$ and $1 \rightarrow 10$.

Fractional Powers

- The generalization to higher powers of words should be clear
- How about rational powers?
- We say a word w is an e 'th power (e rational) if there exist words $y, y' \in \Sigma^*$ such that $w = y^n y'$ and y' is a prefix of y with

$$e = n + \frac{|y'|}{|y|}.$$

- Examples:
 - tormentor is a $\frac{3}{2}$ -power.
 - educated is a $\frac{4}{3}$ -power.
 - onion is a $\frac{5}{3}$ -power.
- A word **avoids e 'th powers** if it contains no subwords that are e' powers for $e' \geq e$.
- A word **avoids e^+ 'th powers** if it contains no subwords that are e' powers for $e' > e$.

Avoiding Large Squares

- As we have seen, it is impossible for infinite binary words to avoid all squares
- But is it possible to avoid arbitrarily large squares?
- Yes! Entringer, Jackson, and Schatz proved in 1974 that there exists an infinite binary word containing no squares yy with $|y| \geq 3$.
- Their strategy: start with any word over three letters that avoids squares, such as

210201210120210201202101210201210120...

Replace each letter by applying the morphism h , as follows:

$$0 \rightarrow 1010$$

$$1 \rightarrow 1100$$

$$2 \rightarrow 0111$$

The resulting word

$$\mathbf{w} = 011111001010011110101100 \dots$$

has the desired properties.

Avoiding Large Squares

- The proof is rather technical, but here are the basic ideas.
- We divide the proof into two cases: w avoids small squares xx , with $3 \leq |x| \leq 8$, and w avoids large squares, $|x| > 8$.
- To see that it avoids small squares, it suffices to check the image of all squarefree strings of length ≤ 5 .
- To see that it avoids large squares, we use the following lemma:

Lemma.

- (a) (inclusion property) Suppose $h(ab) = th(c)u$ for some letters a, b, c and strings t, u . Then this inclusion is trivial (that is, $t = \epsilon$ or $u = \epsilon$).
- (b) (interchange property) Suppose there exist letters a, b, c and strings s, t, u, v such that $h(a) = st$, $h(b) = uv$, and $h(c) = sv$. Then either $a = c$ or $b = c$.

Avoiding Large Squares

- For $i = 1, 2, \dots, n$ define $A_i = h(a_i)$.
- Assume w is squarefree, but $h(w) = xyyz$.
- Write

$$\begin{aligned} h(w) &= A_1 A_2 \cdots A_n \\ &= A'_1 A''_1 A_2 \cdots A_{j-1} A'_j A''_j A_{j+1} \cdots A_{n-1} A'_n A''_n \end{aligned}$$

where $|A''_1|, |A''_j| > 0$ and

$$A_1 = A'_1 A''_1$$

$$A_j = A'_j A''_j$$

$$A_n = A'_n A''_n$$

$$x = A'_1$$

$$y = A''_1 A_2 \cdots A_{j-1} A'_j = A''_j A_{j+1} \cdots A_{n-1} A'_n$$

$$z = A''_n.$$

A'_1	A''_1					A'_j	A''_j				A'_n	A''_n
A_1	A_2	\cdots	A_{j-1}	A_j	A_{j+1}	\cdots	A_{n-1}	A_n				
x	y				y				z			

Avoiding Large Squares

- If $|A_1''| > |A_j''|$, then $A_{j+1} = h(a_{j+1})$ is a subword of $A_1''A_2$, hence a subword of $A_1A_2 = h(a_1a_2)$.
- Thus we can write $A_{j+2} = A'_{j+2}A''_{j+2}$ with

$$A_1''A_2 = A_j''A_{j+1}A'_{j+2}.$$

$$\begin{array}{l}
 y = \begin{array}{|c|c|} \hline A_1'' & A_2 \\ \hline \end{array} \quad \dots \quad \begin{array}{|c|c|} \hline A_{j-1} & A'_j \\ \hline \end{array} \\
 y = \begin{array}{|c|c|c|} \hline A_j'' & A_{j+1} & A'_{j+2} \\ \hline \end{array} \quad \dots \quad \begin{array}{|c|c|} \hline A_{n-1} & A'_n \\ \hline \end{array}
 \end{array}$$

- By the Lemma about inclusions, either $|A_j''| = 0$, or $|A_1''| = |A_j''|$. Both conclusions are impossible.
- Similar reasoning applies if $|A_1''| < |A_j''|$.

Avoiding Large Squares

- Therefore $|A_1''| = |A_j''|$.
- Hence $A_1'' = A_j''$, $A_2 = A_{j+1}$, \dots , $A_{j-1} = A_{n-1}$, and $A_j' = A_n'$.
- Since h is injective, we have $a_2 = a_{j+1}$, \dots , $a_{j-1} = a_{n-1}$.
- It also follows that $|y|$ is divisible by 4 and $A_j = A_j' A_j'' = A_n' A_1''$.
- But by the Lemma, either (1) $a_j = a_n$ or (2) $a_j = a_1$.
- In the first case, $a_2 \cdots a_{j-1} a_j = a_{j+1} \cdots a_{n-1} a_n$, so w contains the square $(a_2 \cdots a_{j-1} a_j)^2$, a contradiction.
- In the second case, $a_1 \cdots a_{j-1} = a_j a_{j+1} \cdots a_{n-1}$, so w contains the square $(a_1 \cdots a_{j-1})^2$, a contradiction.
- It now follows that if w is squarefree then $h(w)$ avoids squares yy with $|y| \geq 3$.

Avoiding Large Squares

- The fact that 3 is best possible can be proved purely mechanically.
- Given a set of forbidden patterns P , we create a tree T as follows:
 - The root of T is labeled ϵ (the empty string).
 - If a node is labeled w and avoids P , then it is an internal node with two children, where the left child is labeled $w0$ and the right child is labeled $w1$.
 - If it does not avoid P , then it is an external node (or “leaf”).
- No infinite word avoiding P exists if and only if T is finite.
- Breadth-first search can be used to verify that T is finite.

Avoiding Large Squares

- Furthermore, certain parameters of T correspond to information about the finite words avoiding P :
 - the number of leaves n is one more than the number of internal nodes, and so $n - 1$ represents the total number of finite words avoiding P ;
 - if the height of the tree (i.e., the length of the longest path from the root to a leaf) is h , then h is the smallest integer such that there are no words of length $\geq h$ avoiding P ;
 - the internal nodes at depth $h - 1$ gives the all words of maximal length avoiding P ;
- In the case of Entringer-Jackson-Schatz, let P be the set of all squares of length ≥ 2 .
- The resulting tree is finite. It has height 19, and contains 478 leaves. The longest label is 010011000111001101 and its complement.

Avoiding Both Powers and Large Squares

- Dekking considered avoiding both cubes and large squares over $\{0, 1\}$.
- He proved that there exists an infinite binary word avoiding both cubes xxx and squares yy with $|y| \geq 4$.
- Furthermore, the bound 4 is best possible.
- This suggests the following natural problem. For each length $l \geq 1$, determine the fractional exponent e such that
 - There is no infinite binary word simultaneously avoiding squares yy with $|y| \geq l$ and e 'th powers
 - There is an infinite binary word simultaneously avoiding squares yy with $|y| \geq l$ and e^+ 'th powers

Avoiding Both Powers and Large Squares

Summary of Results

minimum length l of square avoided	avoidable power	unavoidable power
2	none	all
3	3^+	3
4, 5, 6	$(5/2)^+$	$5/2$
≥ 7	$(7/3)^+$	$7/3$

- The unavoidability results are proved using the tree-traversal technique
- The avoidability results are proved using a strategy similar to Entringer-Jackson-Schatz: we start with a word over $\{0, 1, 2\}$ avoiding squares, and then replace each symbol by an appropriately-chosen binary string.

Avoiding Both Powers and Large Squares

- To show there is an infinite binary word avoiding 3^+ powers and squares yy with $|y| \geq 3$, we use the map

$$0 \rightarrow 0010111010$$

$$1 \rightarrow 0010101110$$

$$2 \rightarrow 0011101010$$

- To show there is an infinite binary word avoiding $\frac{5}{2}^+$ powers and squares yy with $|y| \geq 4$, we use a map sending each letter to a string of 1560 letters. (!)
- To show there is an infinite binary word avoiding $\frac{7}{3}^+$ powers and squares yy with $|y| \geq 7$, we use a map sending each letter to a string of 252 letters.

Example of the Morphism for Length 7

0 → 001101001011001001101100101101001100101100100110110010110011010
010110010011011001011010011001011001101001101100100110100110010
110100110110010011010010110010011011001011010011001011001101001
101100100110100101100100110110010110011010011001011010011011001

1 → 001101001011001001101100101101001100101100100110110010110011010
011001011010011011001001101001011001101001101100100110100110010
110100110110010011010010110010011011001011010011001011001101001
101100100110100101100100110110010110011010011001011010011011001

2 → 001101001011001001101100101101001100101100110100110110010011010
011001011010011011001001101001011001001101100101101001100101100
100110110010110011010010110010011011001011010011001011001101001
101100100110100101100100110110010110011010011001011010011011001

How Were the Morphisms Found?

- How were these morphisms found?
- In the first case, we iteratively generated all words of length $1, 2, 3, \dots$ (up to some bound) that avoid both 3^+ powers and squares yy with $|y| \geq 3$.
- We then guessed such words were the image of a k -uniform morphism applied to a square-free word over $\{0, 1, 2\}$.
- For values of $k = 2, 3, \dots$, we broke up each word into contiguous blocks of size k , and discarded any word for which there were more than 3 blocks.
- For certain values of k , this procedure eventually resulted in 0 words fitting the criteria.
- At this point we knew a k -uniform morphism cannot work, so we increased k and started over.

How Were the Morphisms Found?

- Eventually a k was found for which the number of such words appeared to increase without bound.
- We then examined the possible sets of 3 k -blocks to see if any satisfied the requirements of the Lemma. This gave our candidate morphism.

Enumerating Words Avoiding Patterns

- We can also count the number of words of length n avoiding squares, overlaps, cubes, etc.
- Restivo and Salemi proved that there are only polynomially many overlap-free binary words of length n .
- The current best bound is $O(n^{1.37})$, due to Lepistö
- Brandenburg proved there are exponentially many cubefree binary words of length n .
- This raises the natural question of Kobayashi (1986): at what exponent $2 < e < 3$ does the number of words avoiding e 'th powers jump from polynomial to exponential?
- The surprising answer is $e = 7/3$. There are only polynomially many words of length n avoiding $\frac{7}{3}$ powers, but exponentially many avoiding $\frac{7}{3}^+$ powers.

The Upper Bound

Decomposition Theorem. Let x be a word avoiding α -powers, with $2 < \alpha \leq 7/3$. Let μ be the Thue-Morse morphism, sending $0 \rightarrow 01$, $1 \rightarrow 10$. Then there exist binary words u, v, y with

$$u, v \in \{\epsilon, 0, 1, 00, 11\}$$

such that $x = u\mu(y)v$.

Corollary. Let $2 < \alpha \leq \frac{7}{3}$. There are $O(n^{\log_2 25}) = O(n^{4.644})$ binary words of length n that avoid α -powers.

Proof. Let $x = x_0$ be a nonempty binary word that is α -power-free, with $2 < \alpha \leq \frac{7}{3}$. Then by the decomposition theorem we can write

$$x_0 = u_1\mu(x_1)v_1$$

with $|u_1|, |v_1| \leq 2$. If $|x_1| \geq 1$, we can repeat the process, writing

$$x_1 = u_2\mu(x_2)v_2.$$

The Upper Bound

Continuing in this fashion, we obtain the decomposition

$$x_i = u_i \mu(x_i) v_i$$

until $|x_{t+1}| = 0$ for some t . Then

$$x_0 = u_1 \mu(u_2) \cdots \mu^{t-1}(u_{t-1}) \mu^t(x_t) \\ \mu^{t-1}(v_{t-1}) \cdots \mu(v_2) v_1.$$

Then from the inequalities

$$1 \leq |x_t| \leq 4$$

and

$$2|x_i| \leq |x_{i-1}| \leq 2|x_i| + 4,$$

for $1 \leq i \leq t$, an easy induction gives

$$2^t \leq |x| \leq 2^{t+3} - 4.$$

Thus $t \leq \log_2 |x| < t + 3$, and so

$$\log_2 |x| - 3 < t \leq \log_2 |x|. \quad (1)$$

The Upper Bound

- There are at most 5 possibilities for each u_i and v_i , and there are at most 22 possibilities for x_t (since $1 \leq |x_t| \leq 4$ and x_t is α -power-free).
- Inequality (1) shows there are at most 3 possibilities for t .
- Letting $n = |x|$, we see there are at most $3 \cdot 22 \cdot 5^{2 \log_2 n} = 66n^{\log_2 25}$ words of length n that avoid α -powers.

Arbitrarily Large Squares

Theorem. Every infinite $\frac{7}{3}$ -power-free binary word contains arbitrarily large squares.

Proof.

- Let w be an infinite $\frac{7}{3}$ -power-free binary word.
- By the decomposition theorem and Eq. (1), any prefix of w of length 2^{n+5} contains $\mu^{n+2}(0)$ as a factor.
- But $\mu^{n+2}(0) = \mu^n(0110)$, so any prefix of length 2^{n+5} contains the square factor xx with $x = \mu^n(1)$.

The Shuffle Problem

- Prodinge and Urbanek in 1983 studied the avoidance of arbitrarily large squares in binary words.
- They were unable to answer the following question: is there a pair of infinite binary words, avoiding arbitrarily large squares, such that their perfect shuffle contains arbitrarily large squares?
- Here, by the perfect shuffle of two infinite words $w = a_0a_1a_2 \cdots$ and $x = b_0b_1b_2 \cdots$ we mean the word

$$a_0b_0a_1b_1a_2b_2 \cdots .$$

The Shuffle Problem

The answer is yes. Consider the morphism f defined by

$$\begin{aligned}0 &\rightarrow 001 \\1 &\rightarrow 110.\end{aligned}$$

The fixed point

$$f^\omega(0) = 001001110001001110110110 \dots$$

begins with arbitrarily large squares of the form $f^n(0)f^n(0)$. It is the shuffle of two words

$$0\ 1\ 0\ 1\ 0\ 0\ \dots$$

and

$$0\ 0\ 1\ 1\ 0\ 1\ \dots$$

each of which avoids squares xx with $|x| \geq 4$.

For Further Reading

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