Automaticity

Jeffrey Shallit
Department of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1
Canada
shallit@graceland.uwaterloo.ca
http://math.uwaterloo.ca/~shallit

This talk represents joint work with Y. Breitbart, I. Glaister, J. M. Robson, and C. Pomerance.

Problem:

Given an object (finite string, infinite string, language, etc.), assign a measure of its complexity.

One Solution:

- Kolmogorov–(Chaitin–Solomonoff) complexity
  - See, for example, the recent book by Ming Li and Paul Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications*
- "The complexity of an object is the length of the shortest program to produce it."
- very powerful idea
- lots of applications
- unfortunately not computable!
- is there a computable analogue?

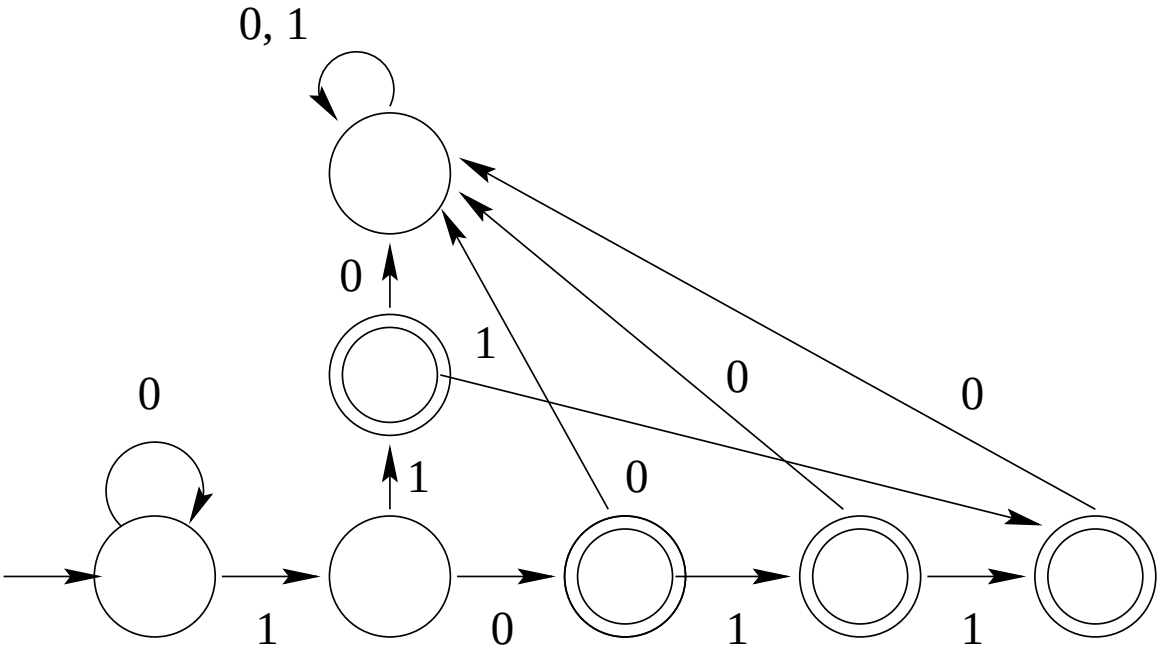
Automaticity

- idea: like Kolmogorov complexity, but replace "Turing machine" with finite automaton
- goals:
  - measure complexity of languages
  - complexity measure is a function
  - regular languages should have $O(1)$ automaticity
  - languages "close" to regular should have "small" automaticity

Basics of Finite Automata

- If Σ is a finite set of symbols, then by Σ^* we mean the free monoid over Σ (set of all finite strings of symbols chosen from Σ plus concatenation as the monoid operation);
- A *language* is a subset of Σ^*.
- a *finite automaton* is a simple model of a computer
- formally it is a quintuple: $M = (Q, \Sigma, \delta, q_0, F)$ where:
  - Q is a finite set of *states*;
  - the *size* of M is $|M| := |Q|$, the number of states;
  - Σ is a finite set of symbols, called the *input alphabet*;
  - $q_0 \in Q$ is the *initial state*;
  - $F \subseteq Q$ is the set of *final states*;
  - $\delta : Q \times \Sigma \to Q$ is the *transition function*
- The *language accepted by* M is denoted by $L(M)$ and is given by $\{w \in \Sigma^* \mid \delta(q_0, w) \in F\}$.

Example of a Finite Automaton



Automaton accepting the base-2 representations of the primes p where $2 \leq p \leq 11$

Automaticity Defined

- $\Sigma^{\leq n} = \epsilon + \Sigma + \Sigma^2 + \cdots + \Sigma^{\leq n}$, the set of all strings in Σ^* of length $\leq n$.
- a language $L \subseteq \Sigma^*$ is an n'th order approximation to a language L' if $L \cap \Sigma^{\leq n} = L' \cap \Sigma^{\leq n}$.
- DFA = class of all deterministic (complete) finite automata over a finite alphabet Σ
- automaticity of a language L is the function which counts the number of states in the smallest DFA that accepts some n'th order approximation to L
- Formally, we define the automaticity of a language L to be the function $A_L(n) =$

  $\min\{|M| \ : \ M \in \mathrm{DFA} \ \text{ and } L(M) \cap \Sigma^{\leq n} = L \cap \Sigma^{\leq n}\}.$

Basic Properties of Automaticity

1. $A_L(n) \leq A_L(n+1)$.
2. L is regular iff $A_L(n) = O(1)$.
3. $A_L(n) = A_{\overline{L}}(n)$.
4. $A_L(n) \leq 2 + \Sigma_{w \in L \cap \Sigma^{\leq n}} |w|$.

Definition. Two strings w, w' are called *n–dissimilar for L* if there exists a string v with $|wv|, |w'v| \leq n$ and either

(i) $wv \in L$, $w'v \notin L$; or

(ii) $wv \notin L$, $w'v \in L$.

Theorem. $A_L(n) =$ the maximum number of distinct pairwise n–dissimilar strings for L.

An Example

Consider the language

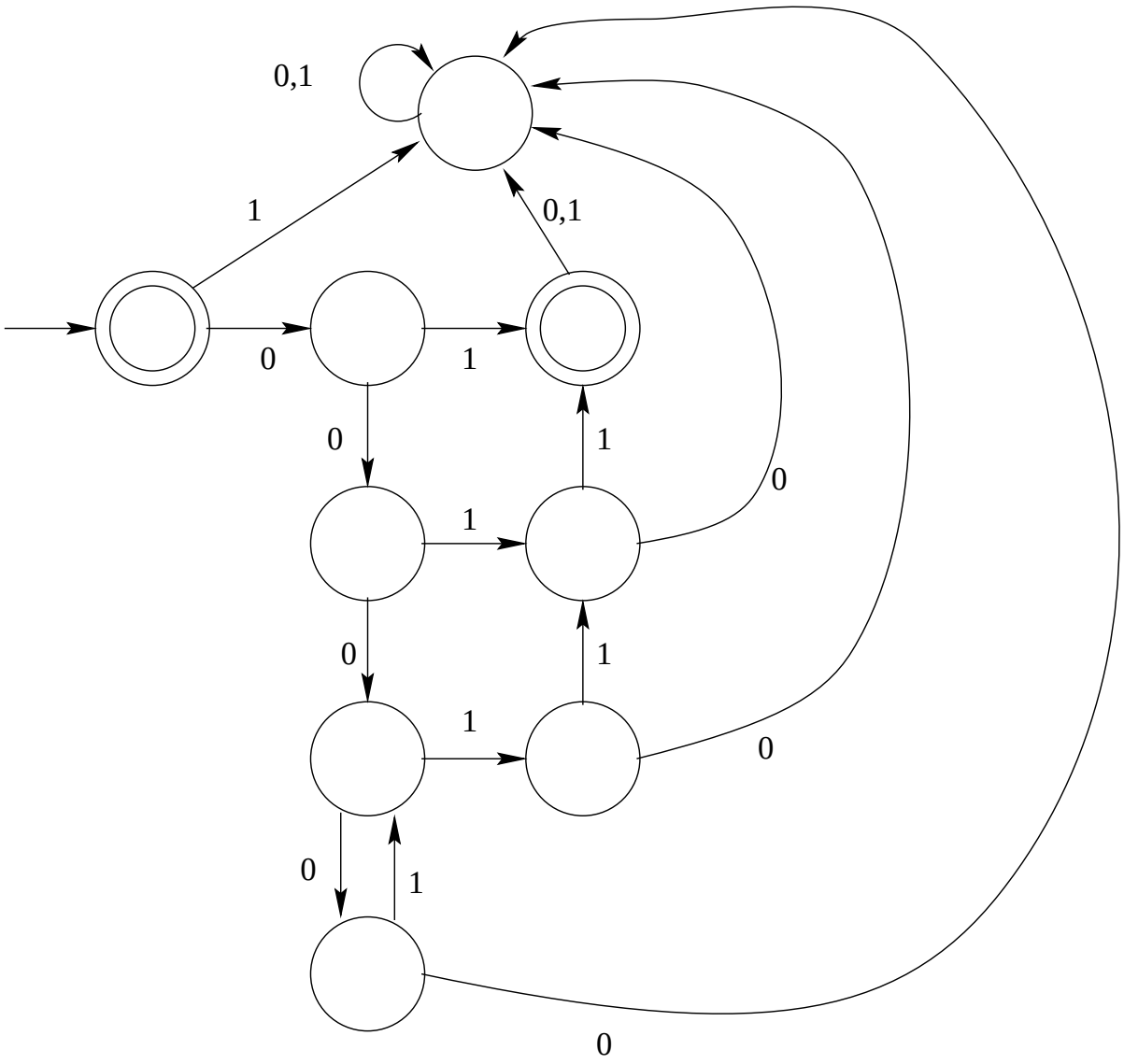
$$L = \{0^n 1^n \ : \ n \geq 0\}.$$

This language is clearly not regular. What is its automaticity?

Automaticity of $L = \{0^n 1^n \ : \ n \geq 0\}$

The automaticity of L is $A_L(n) = 2\lfloor n/2 \rfloor + 1$ for $n \geq 2$.

To see the upper bound, note that we can accept an n'th order approximation to L (for $n = 9$) with the following DFA:



Automaticity of $L = \{0^n 1^n \ : \ n \geq 0\}$

To get the lower bound for $n = 9$, note that we may take

$$\{\epsilon,\ 0,\ 00,\ 000,\ 0000,\ 1,\ 01,\ 001,\ 0001\}$$

as our set of n-dissimilar strings.

This easily generalizes to larger n.

Another Example

Consider

$$P = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, \ldots\},$$

the set of primes expressed in base-2.

A classical theorem due to Minsky and Papert (1966) shows that P is not a regular language.

Theorem. The automaticity of P^R is $\Omega(2^{n/43})$.

(Here P^R denotes the reversal of the set P, i.e. the primes expressed with least significant digit first.)

The basic idea is to prove the following

Lemma. Given integers r, a, b with $r \geq 2$, $1 \leq a, b < r$ with $\gcd(r, a) = \gcd(r, b) = 1$, and $a \neq b$, there exists $m = O(r^{165/4})$ such that $rm + a$ is prime and $rm + b$ is composite.

The proof of this lemma is an easy consequence of a deep theorem of Heath-Brown on the distribution of primes in arithmetic progressions ("Linnik's Theorem").

Taking $r = 2^n$, the lemma implies that there are at least $2^{n/43}$ n-dissimilar strings for the language P^R.

Previous Work

- Trakhtenbrot (1964)
  - called automaticity the "weight of a finite tree"
  - used a different model of automaton
  - proved upper and lower bounds
- Grinberg & Korshunov (1966)
  - improved bounds of Trakhtenbrot
- Karp (1967)
  - introduced (but did not name) the concept of automaticity exactly as we defined it
  - proved that if L is not a regular language, then $A_L(n) \geq (n+3)/2$ for infinitely many n
  - observed that the constants 2 and 3 in this theorem are best possible (but proof was flawed)

- Breitbart (1970, 1971, 1973)
  - studied the automaticity of the language $L_{k,p} = \{w \ : \ w$ is the base–p representation of a perfect k'th power $\}$.
  - gave examples of languages with high automaticity
- Paredaens & Vyncke (1977); Gabarró (1983); Balcázar, Díaz, & Gabarró (1985); Serna (1989)
  - studied similar notions
- Dwork and Stockmeyer (1989)
  - introduced "measure of nonregularity" based on n-dissimilar strings.
  - proved: if a language L is recognized by a two-way probabilistic finite automaton with probability $1/2 + \delta$ $(\delta > 0)$ in time $c^{n^{o(1)}}$, then L is regular.

- Kaneps & Freivalds (1990)
  - They defined $r_{\mathrm{sim}}(L, \Sigma^{\leq n})$ = same definition as Dwork & Stockmeyer for n–dissimilar strings
  - $r_{\mathrm{sim}}(L, \Sigma^{\leq n}) = A_L(n)$
  - independently proved Karp's theorem
  - proved: if a language L is recognized by a probabilistic TM with probability $1/2 + \delta$ for some $\delta > 0$ in $O(\log \log n)$, then L is regular
- Koskas and de Mathan (work in progress, 1996)
  - Showed how to apply automaticity to obtain irrationality measures in finite characteristic

Bounds on Automaticity

Theorem. Let $k = |\Sigma|$ and $C = 1/((\log_2 k)(k-1)^2)$. Then

$$A_L(n) \leq \frac{Ck^{n+2}}{n}(1 + o(1)).$$

Theorem. Let $k = |\Sigma|$. Then for "almost all" languages $L \subseteq \Sigma^*$, we have

$$A_L(n) > (1 - \epsilon)\frac{Ck^{n+1}}{n}$$

for any fixed $\epsilon > 0$ and all sufficiently large n.

Karp's Theorem

Theorem. Let $L \subseteq \Sigma^*$ be a nonregular language. Then

$$A_L(n) \geq (n+3)/2$$

for infinitely many n.

Theorem. The constants 3 and 2 in Karp's theorem are best possible, in the sense that the theorem would be false if 2 were replaced with any smaller number, or if 3 were replaced with any larger number.

Automaticity: The Unary Case

- $k = |\Sigma| = 1$
- different issues arise
- we have $A_L(n) \leq n + 1$, for all L and for all n.

Theorem. Let $L \subseteq 0^*$. Then

$$A_L(n) \leq n + 1 - \lfloor \log_2 n \rfloor$$

for infinitely many n.

Theorem. Let $L \subseteq 0^*$. Then for "almost all" L we have

$$A_L(n) > n - 2 \log_2 n - 2 \log_2 \log_2 n$$

for all sufficiently large n.

Can Karp's Theorem be Improved in the Unary Case?

Recall that Karp proved that if L is not regular, then $A_L(n) \geq (n+3)/2$ infinitely often. This implies that

$$\limsup_{n\to\infty} \frac{A_L(n)}{n} \geq \frac{1}{2}$$

for all nonregular L.

Conjecture. If $L \subseteq 0^*$ is not regular, then

$$\limsup_{n\to\infty} \frac{A_L(n)}{n} \geq \frac{\sqrt{5}-1}{2} \doteq .61803.$$

This conjecture can be expressed in the following equivalent formulation:

An Equivalent Conjecture

Conjecture. Let $w = w_0 w_1 w_2 \cdots$ be an infinite word over a finite alphabet that is not ultimately periodic. Define $s_w(n)$ to be the length of the longest suffix of

$$w_0 w_1 w_2 \cdots w_n$$

that also appears as a subword of $w_0 w_1 w_2 \cdots w_{n-1}$. Then

$$\liminf_{n \to \infty} \frac{s_w(n)}{n} \leq \frac{3 - \sqrt{5}}{2} \doteq .38197.$$

The upper bound is attained for the infinite Fibonacci word (discussed below).

This conjecture has been open since 1994.

The Fibonacci Language

Define $h_1 = 1$, $h_2 = 0$, and $h_{n+1} = h_n h_{n-1}$ for $n \geq 2$. Then

$$\begin{aligned} h_3 &= 01 \\ h_4 &= 010 \\ h_5 &= 01001 \\ h_6 &= 01001010 \\ h_7 &= 0100101001001 \\ &\vdots \end{aligned}$$

Now define

$$\lim_{n\to\infty} h_n = f = f_0 f_1 f_2 \ldots;$$

this is the infinite "Fibonacci word". Let

$$L_f = \{0^i \ : f_i = 0\} = \{\epsilon, 0^2, 0^3, 0^5, 0^7, 0^8, \ldots\}.$$

Theorem.

$$\limsup_{n\to\infty} \frac{A_{L_f}(n)}{n} = \frac{\sqrt{5}-1}{2}.$$

Nondeterministic Automaticity

Let NFA = class of all nondeterministic finite automata.

A *nondeterministic finite automaton (NFA)* is like a deterministic one, except now there can be $0, 1, 2$, or more arrows with the same label leaving any state. A string w is accepted by an NFA if there exists some path labeled w from the initial state to some final state.

The function $N_L(n)$ is the nondeterministic automaticity of the language L, where $N_L(n) =$
$\min\{|M| \ : \ M \in \mathrm{NFA} \ \text{ and } L(M) \cap \Sigma^{\leq n} = L \cap \Sigma^{\leq n}\}$.

Theorem. Let $k = |\Sigma| \geq 2$ and $L \subseteq \Sigma^*$. Then $N_L(n) = O(k^{n/2})$.

Theorem. Let $k = |\Sigma| \geq 2$, and let $L \subseteq \Sigma^*$. Then for almost all L and every $\epsilon > 0$ we have

$$N_L(n) > (1 - \epsilon)k^{n/2}/\sqrt{k-1}$$

for all sufficiently large n.

Lower Bounds for Nondeterministic Automaticity

Theorem. Suppose $L \subseteq \Sigma^*$. If L is not regular, then $N_L(n) \geq \log_2((n+3)/2)$ for infinitely many n.

This lower bound is best possible, up to a constant, since the Stearns-Hartmanis-Lewis language

$$\{2\ (1)_2^R\ 2\ (2)_2^R\ 2\ (3)_2^R\ 2\ (4)_2^R\ 2 \cdots\ 2\ (n)_2^R : n \geq 1\}$$

has nondeterministic automaticity $O(\log n)$. Here $(k)_2$ is the representation of k in base-2, and w^R denotes the reversal of the string w.

Another Example of a Language with Low Nondeterministic Automaticity

Theorem. Define

$$L = \{w \in (0+1)^* \ : \ |w|_0 \neq |w|_1\}.$$

Then L is nonregular and

$$N_L(n) = O((\log n)^2/(\log \log n)).$$

Proof. We need the following fact from number theory:

> Let $n \geq 2$ and suppose $0 \leq i, j < n$. Then $i \neq j$ iff there exists a prime $p \leq 4.4 \log n$ such that $i \not\equiv j \pmod{p}$.

Thus, to nondeterministically accept some nth order approximation to L, we can

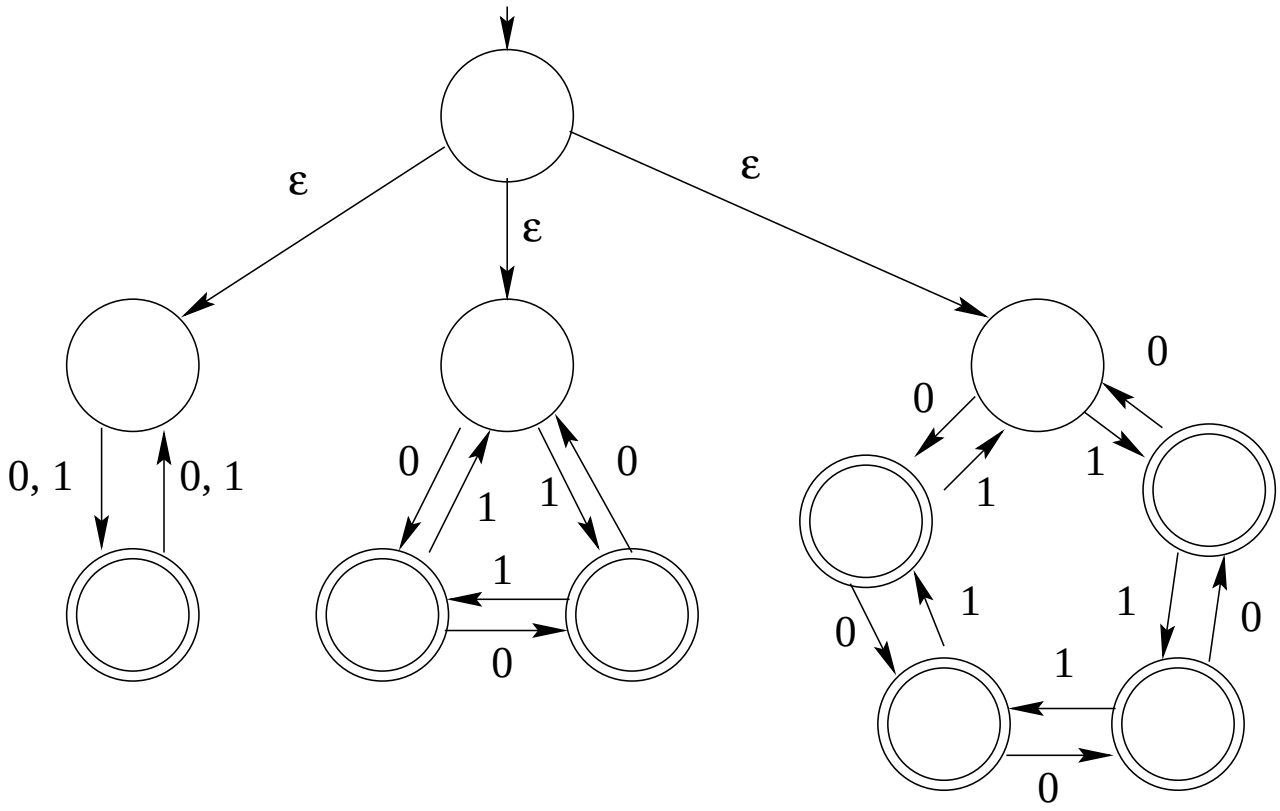
- "guess" the correct prime $p \leq 4.4 \log n$;
- "verify" that $|w|_0 \not\equiv |w|_1 \pmod{p}$.

This construction uses at most

$$1 + \sum_{p \leq 4.4 \log n} p = O((\log n)^2/(\log \log n))$$

states.

Example of Construction



Lower Bounds for Nondeterm. Automaticity
The Unary Case

Theorem. There exists a constant c (which does not depend on L) such that if $L \subseteq 0^*$ is not regular, then

$$N_L(n) \geq c(\log n)^2/(\log \log n)$$

infinitely often.

Open Question: Is this lower bound best possible?

Carl Pomerance has shown that for all monotonically increasing functions f, there exists a language $L = L(f)$ such that

$$N_L(n) = O(f(n)(\log n)^2/(\log \log n)),$$

thus showing the lower bound is essentially tight.

Low Nondeterministic Automaticity: A Unary Example

Theorem. Define $L = \{0^n : n \geq 1$ and the least positive integer not dividing n is not a power of $2\}$. Then L is nonregular and

$$N_L(n) = O((\log n)^3/(\log \log n)).$$

Proof. The construction depends on the following two facts:

> If $0^n \in L$, then there exists a prime power p^k, $p \geq 3$, $k \geq 1$, $p^k \leq 5 \log n$, such that $n \not\equiv 0 \pmod{p^k}$, and $n \equiv 0 \pmod{2^s}$, with $2^s < p^k < 2^{s+1}$.
>
> Further, if such a prime power p^k exists, then $0^n \in L$.

An NFA accepting an n-th order approximation to L can now be constructed as follows:

Low Nondeterministic Automaticity: A Unary Example

- "guess" the correct odd prime power $p^k \leq 5 \log n$;
- verify that, on input 0^r, we have
  - $* \; r \not\equiv 0 \pmod{p^k}$
  - $* \; r \equiv 0 \pmod{2^s}$, with $2^s < p^k < 2^{s+1}$.

This construction uses at most $O((\log n)^3/(\log \log n))$ states.

Polynomial Automaticity

Define the following three complexity classes:

- deterministic polynomial automaticity, or DPA

$$\mathrm{DPA} = \{L \subseteq \Sigma^* \ : \ \exists\, k \text{ such that } A_L(n) = O(n^k)\}.$$

- nondeterministic polynomial automaticity, or NPA

$$\mathrm{NPA} = \{L \subseteq \Sigma^* \ : \ \exists\, k \text{ such that } N_L(n) = O(n^k)\}.$$

- nondeterministic poly-log automaticity, or NPLA

$$\mathrm{NPLA} = \{L \subseteq \Sigma^* \ : \ \exists\, k \text{ such that } N_L(n) = O((\log n)^k)\}.$$

What are the closure properties of these classes, and how are they related?

A Hierarchy of Polynomial Automaticity

Theorem. For all integers $k \geq 0$ there exists a language L_k such that $A_{L_k} = \Theta(n^k)$.

Proof. Let

$$L_k = \{0^{a_1}\ 1\ 0^{a_2}\ 1\ \cdots\ 0^{a_k}\ 1\ \ 0^{a_1}\ 1\ 0^{a_2}\ 1\ \cdots\ 0^{a_k}\ 1\ :$$
$$a_1, a_2, \ldots, a_k \geq 0\}.$$

Theorem. The class DPA is closed under union, intersection, complement, and inverse homomorphism.

Proof. Simply adapt the usual constructions.

Theorem. The class DPA is not closed under concatenation.

Proof. Let $L_1 = (0+1)^*$ and $L_2 = \{1(0+1)^{2^k}\ :\ k \geq 0\}$. Then $A_{L_1}(n) = O(1)$, and $A_{L_2}(n) = O(n)$, but it can be shown that $A_L(n) \geq 2^{n/3}$ for infinitely many n.

An Open Question

Open Question. Is $\mathrm{NPLA} \subseteq \mathrm{DPA}$?

For Further Reading

1. J. Shallit and Y. Breitbart, Automaticity: Properties of a measure of descriptional complexity, in *STACS '94*, Lecture Notes in Comp. Sci. # 775, pp. 619–630. Revised version, to appear, *J. Comput. System Sci.*

2. C. Pomerance and J. M. Robson and J. Shallit, Automaticity II: Descriptional complexity in the unary case, to appear, *Theoret. Comput. Sci.*

3. I. Glaister and J. Shallit, Automaticity III: Polynomial automaticity and context-free languages, to appear, Proceedings of MFCS '96. To appear, *Computational Complexity*.

4. J. Shallit, Automaticity IV: Sequences, Sets, and Diversity, to appear, *J. de Théorie des Nombres de Bordeaux*.