Results on Infinite Words Obtained with an Automatic Prover

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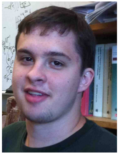
  

The outline

1. Hilbert's dream
2. Fibonacci (Zeckendorf) representation
3. Fibonacci-automatic words
4. Logic
5. The infinite and finite Fibonacci words
6. Fibonacci-regular sequences and enumeration
7. Recent applications to avoidability

1. Hilbert's dream

- To show that every true statement is provable (killed by Gödel)
- To provide an algorithm to prove every provable statement (killed by Turing)
- Nevertheless, some subclasses of problems are decidable - an algorithm will prove or disprove them (e.g., Presburger arithmetic, first-order theory of the reals)
- Another interesting subclass: the Wilf-Zeilberger method for proving binomial coefficient identities

2. Fibonacci (Zeckendorf) representation

- Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

- In analogy with base-2 representation, we can represent every non-negative integer in the form

$$\sum_{0 \le i \le t} \epsilon_i F_{i+2} \quad \text{with} \quad \epsilon_i \in \{0, 1\}.$$

Fibonacci (Zeckendorf) representation

- But then some integers have multiple representations, e.g., $13 = 8 + 5 = 8 + 3 + 2$
- So we impose the additional condition that $\epsilon_i \epsilon_{i+1} = 0$ for all i
- Usually we write the representation in the form

$$\epsilon_t \epsilon_{t-1} \cdots \epsilon_0,$$

with most significant digit first. So, for example, 19 is represented by 101001. This is called Fibonacci or Zeckendorf representation.

Figure: Edouard Zeckendorf (1901–1983)

3. Fibonacci-automatic infinite words

- Consider a finite automaton that takes Fibonacci representation of n as input
- Outputs are associated with the last state reached
- Invalid representations are ignored
- An infinite word results from feeding the representation of each $n \geq 0$ into the automaton
- Example: the Fibonacci infinite word

$$\mathbf{f} = 0100101001001 \cdots$$

Patterns in infinite words

- Squares: words of the form xx (like the French word `couscous`)
- Overlaps: just a little more than a square (like the French word `entente`)
- Cubes: words of the form xxx (like the French sort-of-word `blablabla`)

4. Logic

- Let $\mathrm{Th}(\mathbb{N}, +, 0, 1, <)$ denote the set of all true first-order sentences in the logical theory of the natural numbers with addition.
- Example: in this theory we can express the so-called "Chicken McNuggets theorem" that 43 is the largest integer that cannot be represented as a non-negative integer linear combination of $6, 9$, and 20, as follows:

$$\begin{aligned}(\forall n > 43\ \exists x, y, z \geq 0 \text{ such that } n = 6x + 9y + 20z) \wedge \\ \neg(\exists x, y, z \geq 0 \text{ such that } 43 = 6x + 9y + 20z). \quad (1)\end{aligned}$$

Here, of course, "$6x$" is shorthand for the expression "$x + x + x + x + x + x$", and similarly for $9y$ and $20z$.

Presburger's theorem

Figure: Mojżesz Presburger (1904–1943)

Presburger proved that $\mathrm{Th}(\mathbb{N}, +, 0, 1, <)$ is *decidable*: that is, there exists an algorithm that, given a sentence in the theory, will decide its truth.

Decidability of Presburger arithmetic: proof sketch

- represent integers in an integer base $k \geq 2$ using the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$.
- represent n-tuples of integers as words over the alphabet Σ_k^n, padding with leading zeroes, if necessary. This corresponds to reading the base-k representations of the n-tuples *in parallel*.
- For example, the pair $(21, 7)$ can be represented in base 2 by the word

$$[1, 0][0, 0][1, 1][0, 1][1, 1].$$

Decidability of Presburger arithmetic

- Then the relation $x + y = z$ can be checked by a simple 2-state automaton depicted below, where transitions not depicted lead to a nonaccepting "dead state".

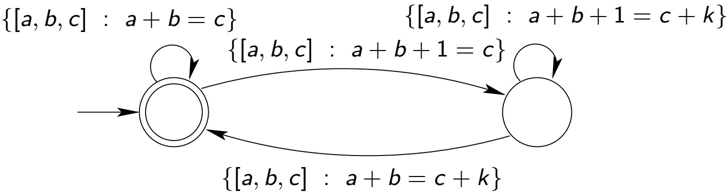


Figure: Checking addition in base k

Decidability of Presburger arithmetic: proof sketch

- Relations like $x = y$ and $x < y$ can be checked similarly.
- Given a formula with free variables $x_1, x_2, \ldots, x_n$, we construct an automaton accepting the base-k expansion of those n-tuples $(x_1, \ldots, x_n)$ for which the proposition holds.
- If a formula is of the form $\exists x_1, x_2, \ldots x_n \; p(x_1, \ldots, x_n)$, then we use nondeterminism to "guess" the x_i and check them.
- If the formula is of the form $\forall p$, we use the equivalence $\forall p \equiv \neg\exists\neg p$; this may require using the subset construction to convert an NFA to a DFA and then flipping the "finality" of states.
- Finally, the truth of a formula can be checked by using the usual depth-first search techniques to see if any final state is reachable from the start state.

The bad news

- The worst-case running time of the algorithm above is bounded above by
$$2^{2^{\cdot^{\cdot^{\cdot^{2^{p(N)}}}}}},$$
where the number of 2's in the exponent is equal to the number of quantifiers, p is a polynomial, and N is the number of states needed to describe the underlying automatic sequence.

The good news

- With a small addition to the logical theory, we can also verify many other kinds of statements (e.g., about the Fibonacci-automatic words)
- Despite bad worst-case bound on running time, an implementation often succeeds in verifying statements in the theory
- Many old results have been verified with this technique, and many new ones proved.

The Fibonacci decision procedure

- Like before, except now all integers are represented in Fibonacci representation
- Comparison is easy
- Addition is harder; need an adder
- There is a 17-state automaton that on input (x, y, z) in Fibonacci representation will determine whether $x + y = z$
- Based on ideas originally due to Berstel

Predicates for various properties

The Fibonacci infinite word $\mathbf{f}$ has a square of length $2j$ beginning at position i:

$$\forall t\ (0 \le t < j) \implies \mathbf{f}[i+t] = \mathbf{f}[i+j+t]$$

This particular word never occurred previously in $\mathbf{f}$:

$$\forall \ell\ (\forall t\ ((0 \le t < 2j)\ \wedge\ \mathbf{f}[i+t] = \mathbf{f}[\ell+t]) \implies \ell \ge i)$$

And this particular word appears in the prefix of length n of $\mathbf{f}$:

$$i + 2j < n$$

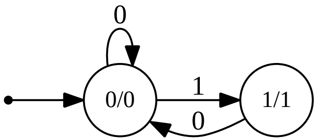
Joining these all with "AND" gives a predicate for triples (n, i, j) such that $\mathbf{f}[i..i+2j-1]$ is a novel square occurring in the prefix of length n in $\mathbf{f}$.

5. The infinite Fibonacci word

The most famous Fibonacci-automatic word is the Fibonacci word

$$\mathbf{f} = 0100101001001010010100100101001001 \cdots ,$$

which can be defined in various ways.
One way is the fixed point of the morphism $\varphi(0) = 01$, $\varphi(1) = 0$.
Another way is the automaton



The infinite Fibonacci word

Yet another way is through the iteration

$$X_1 = 1$$
$$X_2 = 0$$
$$X_n = X_{n-1}X_{n-2}$$

Note that $|X_n| = F_n$.
The $(X_n)_{n\geq 1}$ are called the finite Fibonacci words and for $n \geq 2$ they are all prefixes of $\mathbf{f}$.

The Fibonacci word

Properties of the Fibonacci word have been widely studied.

- $\mathbf{f}$ is not ultimately periodic
- $\mathbf{f}$ contains no 4th powers (Karhumäki, 1983)
- All squares in $\mathbf{f}$ are of order F_n for $n \geq 2$, and squares of all these lengths exist (Séébold, 1985)
- There exist palindromes of all lengths in $\mathbf{f}$ (Chuan, 1993)

All of these claims can easily be verified using our method.

Theorems about the finite Fibonacci words

- Since every finite Fibonacci word is a prefix of length F_n of the infinite Fibonacci word, we can rephrase many claims about the finite Fibonacci words in terms of our logical language
- There are two possible approaches: we can state these claims for length-n prefixes and ask for which n they are satisfied
- Or we can additionally restrict n in our logical language to have Fibonacci representation of the form 10^*

Example claim about the finite Fibonacci words

To illustrate this idea, consider one of the most famous properties of the Fibonacci words, the *almost-commutative* property: letting $\eta(a_1 a_2 \cdots a_n) = a_1 a_2 \cdots a_{n-2} a_n a_{n-1}$ be the map that interchanges the last two letters of a string of length at least 2, we have

Theorem

$X_{n-1} X_n = \eta(X_n X_{n-1})$ for $n \geq 2$.

We can verify this, and prove even more, using our method.

Theorem

Let $x = \mathbf{f}[0..i-1]$ and $y = \mathbf{f}[0..j-1]$ for $i > j > 1$. Then $xy = \eta(yx)$ if and only if $i = F_n$, $j = F_{n-1}$ for $n \geq 3$.

Proof of the almost-commutative property

Proof.

The idea is to check, for each $i > j > 1$, whether

$$\mathbf{f}[0..i-1]\mathbf{f}[0..j-1] = \eta(\mathbf{f}[0..j-1]\mathbf{f}[0..i-1]).$$

We can do this with the following predicate:

$$(i > j) \ \wedge \ (j \geq 2) \ \wedge \ (\forall t, \ j \leq t < i, \ \mathbf{f}[t] = \mathbf{f}[t-j]) \wedge$$
$$(\forall s \leq j{-}3 \ \mathbf{f}[s] = \mathbf{f}[s{+}i{-}j]) \wedge (\mathbf{f}[j{-}2] = \mathbf{f}[i{-}1]) \wedge (\mathbf{f}[j{-}1] = \mathbf{f}[i{-}2]).$$

The resulting automaton accepts $[1,0][0,1][0,0]^+$, which corresponds to $i = F_n$, $j = F_{n-1}$ for $n \geq 4$. □

6. Fibonacci-regular words and enumeration

- In many cases we can count the number $T(n)$ of length-n factors of a Fibonacci-automatic sequence having a particular property P.
- Here by "count" we mean, give an algorithm A to compute $T(n)$ efficiently, that is, in time bounded by a polynomial in $\log n$.
- Although *finding* the algorithm A may not be particularly efficient, once we have it, we can compute $T(n)$ quickly.

Reproving (and fixing) a result of Fraenkel and Simpson

We turn to a result of Fraenkel and Simpson (1999). They computed the exact number of occurrences of all squares appearing in the finite Fibonacci words X_n.
To solve this using our approach, we generalize the problem to consider *any* length-n prefix of $\mathbf{f}$.
The total number of square occurrences in $\mathbf{f}[0..n-1]$:

$$L_{\mathrm{dos}} := \{(n,i,j)_F \ : \ i+2j \leq n \text{ and } \mathbf{f}[i..i+j-1] = \mathbf{f}[i+j..i+2j-1]\}.$$

Let $b(n)$ denote the number of occurrences of squares in $\mathbf{f}[0..n-1]$. First, we use our method to find a DFA M accepting L_{dos}. This (incomplete) DFA has 27 states.

Reproving (and fixing) a result of Fraenkel and Simpson

Next, we compute matrices M_0 and M_1, indexed by states of M, such that $(M_a)_{k,l}$ counts the number of edges (corresponding to the variables i and j) from state k to state l on the digit a of n. We also compute a vector u corresponding to the initial state of M and a vector v corresponding to the final states of M. This gives us the following linear representation of the sequence $b(n)$: if $x = a_1 a_2 \cdots a_t$ is the Fibonacci representation of n, then

$$b(n) = u M_{a_1} \cdots M_{a_t} v, \tag{2}$$

which, incidentally, gives a fast algorithm for computing $b(n)$ for any n.

Reproving (and fixing) a result of Fraenkel and Simpson

Now let $B(n)$ denote the number of square occurrences in the finite Fibonacci word X_n. This corresponds to considering the Fibonacci representation of the form 10^{n-1}; that is, $B(n+1) = b([10^n]_F)$. The matrix M_0 is the following 27×27 array

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
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\end{bmatrix}$$

Reproving (and fixing) a result of Fraenkel and Simpson

- M_0 has minimal polynomial

$$X^4(X-1)^2(X+1)^2(X^2-X-1)^2.$$

- It follows from the theory of linear recurrences that there are constants $c_1, c_2, \ldots, c_8$ such that

$$B(n+1) = (c_1 n + c_2)\alpha^n + (c_3 n + c_4)\beta^n + c_5 n + c_6 + (c_7 n + c_8)(-1)^n$$

  for $n \geq 3$, where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$ are the roots of $X^2 - X - 1$.

- We can find these constants by computing $B(4), B(5), \ldots, B(11)$ and then solving for the values of the constants $c_1, \ldots, c_8$.

Reproving (and fixing) a result of Fraenkel and Simpson

When we do so, we find

$$c_1 = \frac{2}{5} \qquad c_2 = -\frac{2}{25}\sqrt{5} - 2 \qquad c_3 = \frac{2}{5}$$
$$c_4 = \frac{2}{25}\sqrt{5} - 2 \qquad c_5 = 1 \qquad c_6 = 1$$
$$c_7 = 0 \qquad c_8 = 0$$

A little simplification, using the fact that $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, leads to

Theorem

Let $B(n)$ denote the number of square occurrences in X_n. Then

$$B(n+1) = \frac{4}{5} n F_{n+1} - \frac{2}{5}(n+6)F_n - 4F_{n-1} + n + 1$$

for $n \geq 3$.

This statement corrects a small error in their paper.

In a similar way, we can count the cube occurrences in X_n. Using analysis exactly like the square case, we easily find

Theorem

Let $C(n)$ denote the number of cube occurrences in the Fibonacci word X_n. Then for $n \geq 3$ we have

$$C(n) = (d_1 n + d_2)\alpha^n + (d_3 n + d_4)\beta^n + d_5 n + d_6$$

where

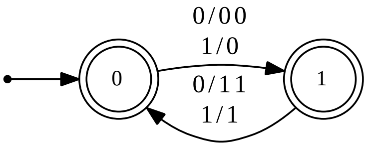
$$d_1 = \frac{3-\sqrt{5}}{10} \qquad d_2 = \frac{17}{50}\sqrt{5} - \frac{3}{2}$$
$$d_3 = \frac{3+\sqrt{5}}{10} \qquad d_4 = -\frac{17}{50}\sqrt{5} - \frac{3}{2}$$
$$d_5 = 1 \qquad d_6 = -1.$$

7. Avoidability

- Consider avoiding the pattern xxx^R.
- This pattern occurs, for example, in the French word `choisissiez`, with $x = \texttt{is}$.
- There are periodic infinite binary words avoiding this pattern, like
$$(01)^\omega = 010101\cdots$$
and
$$(0010011011)^\omega = 001001101100100110110010011011\cdots .$$
- But are there infinite aperiodic words avoiding xxx^R?

Avoidability

Yes! Take the infinite Fibonacci word **f** and run it through the following transducer:



obtaining the infinite word

$$\mathbf{r} = 001001101101100100110110110010010011011001001001101100 \cdots .$$

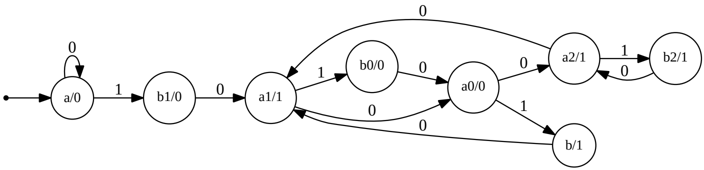
Claim: it avoids the patterns xxx^R and also xx^Rx^R.

Avoiding xxx^R

To prove this we use the predicate

$$\exists i \geq 0 \forall t,\ 0 \leq t < n\, (\mathbf{r}[i+t] = \mathbf{r}[i+t+n]) \wedge (\mathbf{r}[i+t] = \mathbf{r}[i+3n-1-t]),$$

which says that the first block of length n equals the second block, and the first block equals the reverse of the the third block.
The word $\mathbf{r}$ itself is generated by an 8-state automaton:



When we run this predicate on the automaton, we get that only length $n = 0$ is accepted. So the pattern xxx^R doesn't occur. This takes about 8 seconds on a laptop and the largest intermediate automaton has 8304 states.

Other applications of the method

- The Tribonacci-automatic words: based on a recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.
- The paperfolding words: we can prove theorems about uncountably many different sequences simultaneously!
- The Sturmian words: modulo a few details which need to be proven, Luke Schaeffer can show that there is a decidable theory for these words, too.