Abelian and Additive Powers in the Tribonacci Word

Jeffrey Shallit

School of Computer Science, University of Waterloo
Waterloo, ON N2L 3G1 Canada
shallit@uwaterloo.ca
https://cs.uwaterloo.ca/~shallit/

Abelian and additive powers

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A word $x \in \mathbb{N}^*$ is said to be an *additive k-power* if it can be written in the form $x = y_1 y_2 \cdots y_k$ where $|y_i| = |y_1|$ *and* $\sum y_i = \sum y_1$ for all i.

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The length of each y_i is called the *order* of the abelian (resp., additive) power.

The Fibonacci word

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$$\mathbf{f} = 0100101001001010010100 \cdots$$

is the infinite fixed point of the morphism $0 \rightarrow 01$, $1 \rightarrow 0$.

We know from a 2016 paper of Fici, Langiu, Lecroq, Lefebvre, Mignosi, Peltomäki, and Prieur-Gaston that the Fibonacci word has an abelian k-power of order n if and only if $\lfloor k\varphi n \rfloor \equiv 0, -1 \pmod{k}$, where $\varphi = (1+\sqrt{5})/2$, the golden ratio.

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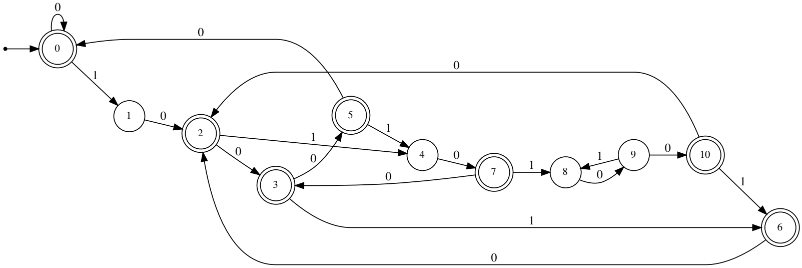
So abelian powers in these words are well understood. For example, OEIS sequence A336487 consists of those n for which there is an abelian cube of order n in $\mathbf{f}$:

$$2, 3, 5, 6, 7, 8, 10, 11, 13, 15, 16, 18, 19, 21, 23, 24, 26, \ldots$$

Fibonacci automaton for abelian cube orders in the Fibonacci word

It turns out that there is an 11-state finite automaton accepting, in Fibonacci representation, exactly those *n* for which there is an abelian cube of order *n* in the Fibonacci word:



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More generally: Charlier, Rampersad, Rigo, and Waxweiler proved in 2011 (among many other things) that the minimal Fibonacci automaton recognizing multiples of k has $2k^2$ states.

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This, together with the criterion of Fici et al. mentioned previously, and the observation that the function $n \to \lfloor n\varphi \rfloor$ is computed by a synchronized Fibonacci DFA of 7 states, shows that the orders of abelian k-powers in $\mathbf{f}$ are recognized by a Fibonacci DFA of $O(k^2)$ states.

The Tribonacci word

Recall: the Tribonacci word

$$\mathbf{tr} = 0102010 \cdots$$

is the fixed point of the morphism

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We are interested in the abelian and additive powers appearing in **tr**.

Tribonacci numbers and Tribonacci representation

Remember: the Tribonacci numbers T_i are defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and

$$T_i = T_{i-1} + T_{i-2} + T_{i-3}$$

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for $i \geq 3$.

Every natural number can be represented uniquely in *Tribonacci representation* as

$$n = \sum_{1 \leq i \leq r} e_i T_{r+2-i}$$

for $e_i \in \{0, 1\}$ provided $e_i e_{i+1} e_{i+2} \neq 1$. We write $(n)_T = e_1 \cdots e_r$.

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Example: $(43)_T = 110110$ because

$$43 = T_7 + T_6 + T_4 + T_3 = 24 + 13 + 4 + 2.$$

Alternative representation for the Tribonacci word

Theorem. The n'th symbol of the Tribonacci word **tr** (starting at index 0) is

$$\begin{cases} 0, & \text{if } (n)_T \text{ ends in } 0; \\ 1, & \text{if } (n)_T \text{ ends in } 01; \\ 2, & \text{if } (n)_T \text{ ends in } 011. \end{cases}$$

This means that the Tribonacci word can be computed by an automaton reading n represented in Tribonacci representation.

Abelian squares in the Tribonacci word

Theorem.

(a) There are abelian squares of all orders in **tr**.

(b) Furthermore, if we consider two abelian squares xx' and yy' to be the same if x is a permutation of y, then every order has either one or two abelian squares.

(c) Both possibilities occur infinitely often.

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(c) Both possibilities occur infinitely often.

Parts (b) and (c) seem to be new.

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Theorem. There is a (minimal) Tribonacci automaton of 1169 (!) states recognizing the Tribonacci representation of those n for which there is an abelian cube of order n in **tr**.

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Theorem. There is a (minimal) Tribonacci automaton of 1169 (!) states recognizing the Tribonacci representation of those *n* for which there is an abelian cube of order *n* in **tr**.

The sequence of these *n* is

$$4, 6, 7, 11, 13, 17, 18, 20, 24, 26, 27, 30, 31, 33, \ldots$$

and is sequence A345717 in the OEIS.

Additive cubes in the Tribonacci word

Theorem. There is a (minimal) Tribonacci automaton of 4927 (!) states recognizing the Tribonacci representation of those n for which there is an additive cube of order n in **tr**.

The sequence of these n is

$$3, 4, 6, 7, 10, 11, 13, 14, 16, 17, 18, 20, 21, 23, 24, 26, 27, 30, 31, 33, \ldots$$

and is sequence A347752 in the OEIS.

How the results are proved

It turns out that the frequency of each letter $0, 1, 2$ in **tr** is *Tribonacci-synchronized*.

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This means that there is a Tribonacci automaton recognizing, in parallel, n and $|\mathbf{tr}[0..n-1]|_i$ for $i \in \{0, 1, 2\}$.

So we can write first-order formulas for any fixed abelian power or additive power in the Tribonacci word, and use the `Walnut` software to create automata for abelian and additive powers.

A research question and a research project

Research question. Is there some simpler description of the orders of abelian and additive cubes in the Tribonacci word?

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Research project. Try to understand the orders of abelian and additive powers in episturmian words. Is there something akin to the result of Fici et al.?