Some Predictable Pierce Expansions

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I. Introduction.

In 1929, T. A. Pierce discussed an algorithm for expanding real numbers \( x \in (0, 1) \) in the form

\[
x = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} \ldots
\]

where the \( a_i \) form a strictly increasing sequence of positive integers.

He showed that these expansions (which we call **Pierce expansions**) are essentially unique. The Pierce expansion for \( x \) terminates if and only if \( x \) is rational. See [Pie] or [Sha] for details.

In this note, we give formulas for the \( a_i \) in the case where

\[
x = \frac{c - \sqrt{c^2 - 4}}{2}
\]

and \( c \geq 3 \) is an integer. For these numbers, Pierce expansions provide extremely rapidly converging series.

II. Finding Real Roots of Polynomials.

To save space, we will sometimes write the equation (1) in the form

\[
x = \{ a_1, a_2, a_3, \ldots \}
\]

where the curly brackets denote a Pierce expansion.

Let

\[
p_i(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0
\]

be a polynomial with integers coefficients and a single real zero \( \alpha \) in the interval \((0, 1)\). We want to find the first term in the Pierce expansion of \( \alpha \). From equation (1) it is easy to see that \( a_1 = \lfloor 1/\alpha \rfloor \). Consider the polynomial \( q_1(x) = x^n p_1(1/x) \); this is a polynomial with integer co-efficients that has \( 1/\alpha \) as a zero. Through a simple binary search procedure, it is easy to find \( d_1 \) such that

\[
\text{sign}(q(d_1)) \neq \text{sign}(q(d_1 + 1));
\]

this shows that \( d_1 = \lfloor 1/\alpha \rfloor \) and so we can take \( \alpha_1 = d_1 \).

Now consider the polynomial
\[ p_2(x) = a_1^n p_1 \left( \frac{1-x}{a_1} \right) \]

This again is a polynomial with integer coefficients. It is easily verified that if \( \beta \) is a zero of \( p_2(x) \) then
\[
\alpha = \frac{1}{a_1} - \frac{1}{a_1} \beta
\]
so
\[
\beta = \frac{1}{a_2} - \frac{1}{a_2} \varepsilon + \ldots
\]

By repeating this procedure on the polynomial \( p_2(x) \), we generate the coefficient \( a_2 \) in the Pierce expansion of \( \alpha \). And by continuing in the same fashion, we can generate as many terms of the Pierce expansion for \( \alpha \) as desired:
\[
\alpha = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \ldots
\]

Now let us specify our polynomial to be
\[
p_1(x) = x^2 - cx + 1
\]
where \( c \geq 3 \) is an integer. Let \( \alpha \) be the smaller positive zero, so
\[
\alpha = \frac{c - \sqrt{c^2 - 4}}{2}. \tag{2}
\]

Now \( q_1(x) = x^2 p_1(1/x) = x^2 - cx + 1 \). We find \( q_1(c-1) = 2 - c \), which is negative, and \( q_1(c) = 1 \) which is positive. Hence we see that \( a_1 = c - 1 \).

Now \( p_2(x) = (c - 1)^2 p_1 \left( \frac{1-x}{c-1} \right) \); hence
\[
p_2(x) = x^2 + (c^2 - c - 2)x + 2 - c.
\]

We find
\[
q_2(x) = x^2 p_2(1/x) = (2 - c)x^2 + (c^2 - c - 2)x + 1
\]

Now \( q_2(c+1) = 1 \) which is positive; but \( q_2(c+2) = 5 - c^2 \) which is negative. Hence we see that \( a_2 = c + 1 \).

Now \( p_3(x) = x^2 p_2 \left( \frac{1-x}{c+1} \right) \) so we see
\[
p_3(x) = x^2 - (c^3 - 3c)x + 1.
\]

So far we have been following the algorithm. But now we notice that \( p_3(x) \) is essentially just \( p_1(x) \) with \( c^3 - 3c \) playing the role of \( c \). We have found
\[
\alpha = \frac{1}{c-1} - \frac{1}{(c-1)(c+1)} + \frac{1}{(c-1)(c+1)} \gamma
\]
where \( \gamma \) is the root of \( x^2 - (c^3 - 3c)x + 1 = 0 \). By continuing this process, we get

Theorem.
Let $\alpha$ be as in equation (2). Then

$$\alpha = \{ c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, c_2 - 1, c_2 + 1, \ldots \}$$

where $c_0 = c$, $c_{k+1} = c_k^3 - 3c_k$.

For example, let $c = 3$. Then we find

$$\frac{3 - \sqrt{5}}{2} \{ 2, 4, 17, 5777, 5779, \ldots \}$$

Another example: let $c = 6$. Then, after some manipulation, we find:

$$\sqrt{2} - 1 = \{ 2, 5, 7, 197, 199, 7761797, 7761799, \ldots \}$$

Ironically, both Pierce and Salzer [Sal] gave the first four terms of this expansion, but apparently neither detected the general pattern!

### III. The Coefficients $c_k$

The recurrence $c_{k+1} = c_k^3 - 3c_k$ is an interesting one which has been previously studied [AhSl], [Esc]. Some brief comments are in order.

If we let $\alpha$ and $\beta$ be the roots of the quadratic

$$x^2 - cx + 1 = 0$$

and define

$$V(n) = \alpha^n + \beta^n; \quad U(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

then it is easy to show by induction that

$$V(n) = cV(n-1) - V(n-2); \quad U(n) = cU(n-1) - U(n-2)$$

where

$$V(0) = 2, \quad V(1) = c; \quad U(0) = 0, \quad U(1) = 1$$

We can also show that $V(3k) = V(k)^3 - 3V(k)$; hence by induction $c_k = V(3^k)$. This gives the following closed form for the $c_k$:

$$c_k = \left( \frac{c + \sqrt{c^2 - 4}}{2} \right)^{3^k} + \left( \frac{c - \sqrt{c^2 - 4}}{2} \right)^{3^k}$$

Similarly, it is easy to show by induction that

$$\frac{U(3^k - 1)}{U(3^k)} = \{ c_0 - 1, c_0 + 1, c_1 - 1, c_1 + 1, \ldots, c_{k-1} - 1, c_{k-1} + 1 \}$$

which gives an alternative proof of our Theorem.
References


