A Simple Proof that Phi is Irrational*

Jeffrey Shallit
Student, Lower Merion High School
Ardmore, Pennsylvania 19003

Most proofs of the irrationality of phi, the golden ratio, involve the concepts of number fields and the irrationality of $\sqrt{5}$. This proof involves only very simple algebraic concepts.

Denoting the golden ratio as ϕ, we have

$$\phi^2 - \phi - 1 = 0.$$

Assume $\phi = p/q$, where p and q are integers with no common factors except 1. For if p and q had a common factor we could divide it out to get a new set of numbers p' and q'.

Then

$$\begin{aligned} (p/q)^2 - (p/q) - 1 &= 0 \\ (p/q)^2 - (p/q) &= 1 \\ p^2 - pq &= q^2 \\ p(p-q) &= q^2 \qquad (1) \end{aligned}$$

Equation (1) implies that p divides q^2, and therefore p and q have a common factor. But we already know that p and q have no common factor other than 1, and p cannot equal 1 because this would imply $q = 1/\phi$, which is not an integer. Therefore our original assumption that $\phi = p/q$ is false and ϕ is irrational.

*From *Fibonacci Quarterly* **13** (1975) 32