# The Mathematics of Per Nørgård's Rhythmic Infinity System 

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## 1 Introduction

The Danish composer Per Nørgård (1932-) invented a procedure for generating rhythms which was described by Erling Kullberg [5]. Reworded in mathematical notation, this procedure is as follows:

Let the Fibonacci numbers $\left(F_{n}\right)_{n \geq 0}$ be defined as usual by $F_{0}=0, F_{1}=1$, and $F_{n}=$ $F_{n-1}+F_{n-2}$. Starting with the pair $\left(c_{0}, c_{1}\right)=\left(F_{2 n}, F_{2 n+1}\right)$, perform the following operation $n-2$ times:

- If a number $F_{i}$ appears in an even-indexed position, replace it with ( $F_{i-2}, F_{i-1}$ )
- If a number $F_{i}$ appears in an odd-indexed position, replace it with ( $F_{i-1}, F_{i-2}$ )

Kullberg illustrates this procedure in the case $n=5$, as follows:


Figure 1: Generating the rhythmic infinity series

[^0]Here, starting with the pair $(55,89)$, we replace 55 by $(21,34)$ and 89 by $(55,34)$ to get the quadruple $(21,34,55,34)$, and so forth.

After $n-2$ iterations, the resulting sequence is of length $2^{n-1}$. As $n \rightarrow \infty$ we get a limiting sequence $\left(a_{i}\right)_{i \geq 0}$ :

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $a_{i}$ | 3 | 5 | 8 | 5 | 8 | 13 | 8 | 5 | 8 | 13 | 21 | 13 | 8 | 13 | 8 | 5 | 8 | 13 | $\cdots$ |

In this paper I obtain an explicit formula for the sequence $\left(a_{i}\right)_{i \geq 0}$ and show how it is related to binary Gray code.

We can see the structure of the sequence $\left(a_{i}\right)_{i \geq 0}$ more easily if we replace each number in Figure 1 by the corresponding Fibonacci number, as follows:


Figure 2: Generating the rhythmic infinity system
This gives us a sequence $\left(b_{i}\right)_{i \geq 0}$ defined by $a_{i}=F_{b_{i}}$ :

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | :--- |
| $b_{i}$ | 4 | 5 | 6 | 5 | 6 | 7 | 6 | 5 | 6 | 7 | 8 | 7 | 6 | 7 | 6 | 5 | 6 | 7 | $\cdots$ |

Finally, if we define $c_{i}=b_{i}-4$, we get the following sequence:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $c_{i}$ | 0 | 1 | 2 | 1 | 2 | 3 | 2 | 1 | 2 | 3 | 4 | 3 | 2 | 3 | 2 | 1 | 2 | 3 | $\cdots$ |

We now find another way to generate the sequence $\left(c_{i}\right)_{i \geq 0}$ : through iterated morphisms.
Let $\Sigma$ be a finite set of symbols, called an alphabet. Then $\Sigma^{*}$ denotes the set of all finite strings with symbols chosen from $\Sigma$. For example,

$$
\{0,1\}^{*}=\{\epsilon, 0,1,00,01,10,11,000, \ldots\}
$$

Here $\epsilon$ is the symbol for the empty string.
A morphism is a map $h: \Sigma^{*} \rightarrow \Sigma^{*}$ that satisfies the identity $h(x y)=h(x) h(y)$ for all strings $x, y \in \Sigma^{*}$. A morphism may be iterated by defining $h^{0}$ to be the identity map (i.e., $h^{0}(x)=x$ for all $\left.x \in \Sigma^{*}\right)$ and $h^{i}(x)=h^{i-1}(h(x))$ for $i \geq 1$.

Iterated morphisms have been used by the composer Tom Johnson in some of his work; for more details see [1, 2].

To generate $\left(c_{i}\right)_{i \geq 0}$ we may model Nørgård's transformation as follows: we define a map

$$
\mu:[a, b] \rightarrow[a-2, a-1][b-1, b-2] .
$$

This map can be extended to a morphism on sequences of pairs using the rule $\mu(x y)=$ $\mu(x) \mu(y)$. Then the first $2^{n-1}$ terms of the sequence $\left(b_{i}\right)_{i \geq 0}$ are given by $\mu^{n-2}([2 n, 2 n+1])$, and the first $2^{n+1}$ terms of the sequence $\left(c_{i}\right)_{i \geq 0}$ are given by $\mu^{n}([2 n, 2 n+1])$.

For example:

$$
\begin{aligned}
\mu^{0}([6,7]) & =[6,7] \\
\mu^{1}([6,7]) & =[4,5][6,5] \\
\mu^{2}([6,7]) & =[2,3][4,3][4,5][4,3] \\
\mu^{3}([6,7]) & =[0,1][2,1][2,3][2,1][2,3][4,3][2,3][2,1]
\end{aligned}
$$

This generates the sequence $\left(c_{i}\right)_{i \geq 0}$ in a "top-down" fashion.
To generate $\left(c_{i}\right)_{i \geq 0}$ in a "bottom-up" fashion we introduce a morphism $\varphi$ defined by

$$
\begin{aligned}
\varphi([a, a+1]) & =[a, a+1][a+2, a+1] \\
\varphi([a+1, a]) & =[a+1, a+2][a+1, a]
\end{aligned}
$$

Theorem 1 For $n \geq 0$ we have

$$
\begin{equation*}
\mu^{n}([2 n, 2 n+1])=\varphi^{n}([0,1]) \tag{1}
\end{equation*}
$$

Proof. It turns out to be useful to prove something more general. Namely, we prove the following two equations simultaneously by mathematical induction on $n$ :

$$
\begin{align*}
\mu^{n}([k, k+1]) & =\varphi^{n}([k-2 n, k+1-2 n]) ;  \tag{2}\\
\mu^{n}([k+1, k]) & =\varphi^{n}([k+1-2 n, k-2 n]) ; \tag{3}
\end{align*}
$$

for all integers $k$.
It is easy to see (2) and (3) hold for $n=0$. Now assume (2) and (3) hold for $n$; we prove them for $n+1$.

$$
\begin{aligned}
\mu^{n+1}([k, k+1]) & =\mu^{n}(\mu([k, k+1])) \\
& =\mu^{n}([k-2, k-1][k, k-1]) \\
& =\mu^{n}([k-2, k-1]) \mu^{n}([k, k-1]) \\
& =\varphi^{n}([k-2-2 n, k-1-2 n]) \varphi^{n}([k-2 n, k-1-2 n]) \\
& =\varphi^{n}([k-2-2 n, k-1-2 n][k-2 n, k-1-2 n]) \\
& =\varphi^{n}(\varphi([k-2-2 n, k-1-2 n])) \\
& =\varphi^{n+1}([k-2(n+1), k+1-2(n+1)]) .
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mu^{n+1}([k+1, k]) & =\mu^{n}(\mu([k+1, k])) \\
& =\mu^{n}([k-1, k][k-1, k-2]) \\
& =\mu^{n}([k-1, k]) \mu^{n}([k-1, k-2]) \\
& =\varphi^{n}([k-1-2 n, k-2 n]) \varphi^{n}([k-1-2 n, k-2-2 n]) \\
& =\varphi^{n}([k-1-2 n, k-2 n][k-1-2 n, k-2-2 n]) \\
& =\varphi^{n}(\varphi([k-1-2 n, k-2-2 n]) \\
& =\varphi^{n+1}([k+1-2(n+1), k-2(n+1)]) .
\end{aligned}
$$

Finally, the desired result (1) follows by setting $k=2 n$ in (2).
It now follows that we can generate the sequence $c_{i}$ by iterating the morphism $\varphi$ starting with $[0,1]$. For example

$$
\begin{aligned}
\varphi^{0}([0,1]) & =[0,1] \\
\varphi^{1}([0,1]) & =[0,1][2,1] \\
\varphi^{2}([0,1]) & =[0,1][2,1][2,3][2,1] \\
\varphi^{3}([0,1]) & =[0,1][2,1][2,3][2,1][2,3][4,3][2,3][2,1] \\
& \vdots
\end{aligned}
$$

As a consequence we get

## Corollary 2

$$
\varphi\left(\left[c_{2 i}, c_{2 i+1}\right]\right)=\left[c_{4 i}, c_{4 i+1}\right]\left[c_{4 i+2}, c_{4 i+3}\right] .
$$

We now introduce the so-called "pattern functions" $e_{P}(n)$. Let $P$ be a string of 0 's and 1's. Then $e_{P}(n)$ counts the number of (possibly overlapping) occurrences of $P$ in the base- 2 expansion of $n$. For example, $\epsilon_{10}(12)=1$, since the base- 2 representation of 12 is 1100 , and this contains one occurrence of 10 .

In the case where $P$ starts with a 0 , some additional elaboration is necessary. In this case we assume that the base- 2 representation of $n$ starts with $|P|-1$ zeroes. For example, $\epsilon_{01}(12)=1$.

We define $d_{n}=e_{01}(n)+e_{10}(n)$. It is easy to see that, for $n>0$, the quantity $d_{n}$ counts the number of distinct blocks of adjacent identical symbols in the binary expansion of $n$. For example, the binary expansion of 399 is 110001111 , which has 3 blocks (namely 11,000 , and 1111). We have $d_{399}=e_{01}(399)+e_{10}(399)=2+1=3$.

Here is a table:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\cdots$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| $\epsilon_{01}(i)$ | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 2 |
| $\cdots$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| $\epsilon_{10}(i)$ | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| $d_{i}$ | 0 | 1 | 2 | 1 | 2 | 3 | 2 | 1 | 2 | 3 | 4 | 3 | 2 | 3 | 2 | 1 | 2 | 3 |
| $d_{i}$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

Theorem 3 We have $c_{n}=d_{n}$ for $n \geq 0$.

Proof. By comparing the binary expansions of $2 n, 2 n+1$ with those of $4 n, 4 n+1,4 n+2$, $4 n+3$, we easily see that

$$
\begin{aligned}
d_{4 n} & =d_{2 n} \\
d_{4 n+1} & =d_{2 n}+1 \\
d_{4 n+2} & =d_{2 n+1}+1 \\
d_{4 n+3} & =d_{2 n+1}
\end{aligned}
$$

for $n \geq 0$. Since $c_{0}=d_{0}=0$, the equality $c_{n}=d_{n}$ for all $n \geq 0$ will follow if we can show that $\left(c_{n}\right)_{n \geq 0}$ satisfies the same relations as those for $d$ given above.

To see this, we consider the case $n$ even and $n$ odd separately.
If $n$ is even, then $c_{2 n+1}=c_{2 n}+1$. Using this fact and Corollary 2, we find

$$
\begin{aligned}
{\left[c_{4 n}, c_{4 n+1}\right]\left[c_{4 n+2}, c_{4 n+3}\right] } & =\varphi\left(\left[c_{2 n}, c_{2 n+1}\right]\right) \\
& =\varphi\left(\left[c_{2 n}, c_{2 n}+1\right]\right) \\
& =\left[c_{2 n}, c_{2 n}+1\right]\left[c_{2 n}+2, c_{2 n}+1\right] \\
& =\left[c_{2 n}, c_{2 n}+1\right]\left[c_{2 n+1}+1, c_{2 n+1}\right],
\end{aligned}
$$

from which the desired relations follow.
If $n$ is odd, then $c_{2 n+1}=c_{2 n}-1$. Using this fact and Corollary 2 again, we find

$$
\begin{aligned}
{\left[c_{4 n}, c_{4 n+1}\right]\left[c_{4 n+2}, c_{4 n+3}\right] } & =\varphi\left(\left[c_{2 n}, c_{2 n+1}\right]\right) \\
& =\varphi\left(\left[c_{2 n}, c_{2 n}-1\right]\right) \\
& =\left[c_{2 n}, c_{2 n}+1\right]\left[c_{2 n}, c_{2 n}-1\right] \\
& =\left[c_{2 n}, c_{2 n}+1\right]\left[c_{2 n+1}+1, c_{2 n+1}\right]
\end{aligned}
$$

from which the desired relations follow.
The sequence $\left(d_{n}\right)_{n \geq 0}$ defined by $d_{n}=\epsilon_{01}(n)+e_{10}(n)$ is well-known: in addition to its characterization as the number of distinct blocks of adjacent identical symbols in the binary expansion of $n$, it is also the sum of the bits in the Gray code representation of $n[4,3]$. From this, the identity $\left|d_{n}-d_{n-1}\right|=1$ for $n \geq 1$ easily follows. This explains its attractiveness as a basis for music composition: the sequence $\left(d_{n}\right)_{n \geq 1}$ makes no large jumps, and hence when used as an index into the Fibonacci numbers it "alternately expands and contracts in a gently undulating form" [5].

We can now prove our closed-form for Nørgård's rhythmic infinity sequence:
Theorem 4 We have $a_{i}=F_{d(i)+4}=F_{\epsilon_{01}(i)+\epsilon_{10}(i)+4}$ for $i \geq 0$.

Proof. We have $c_{i}=d_{i}=\epsilon_{01}(i)+e_{10}(i)$ by Theorem 3. On the other hand, by definition we have $c_{i}=b_{i}-4$ and $a_{i}=F_{b_{i}}$. Putting this all together gives the desired relation for $a_{i}$.

Next we give an additional method of generating the sequence $\left(c_{i}\right)_{i \geq 0}$. Define

$$
\begin{aligned}
X_{n} & =c_{0} c_{1} c_{2} \cdots c_{2^{n}-1} \\
Y_{n} & =c_{2^{n}} c_{2^{n}+1} \cdots c_{2^{n+1}-1}
\end{aligned}
$$

for $n \geq 0$; thus $X_{n}$ and $Y_{n}$ are blocks of $2^{n}$ symbols. Let $X$ be a block of symbols. By $X+a$ we mean the block that results by adding $a$ to each symbol in $X$.

Theorem 5 We have

$$
\begin{aligned}
X_{n+1} & =X_{n} Y_{n} \\
Y_{n+1} & =\left(X_{n}+2\right) Y_{n} .
\end{aligned}
$$

Proof. The result for $X_{n}$ follows immediately from the definition. Thus it suffices to show that

$$
c_{2^{n+1}+a}=c_{a}+2
$$

and

$$
c_{2^{n+1}+2^{n}+a}=c_{2^{n}+a}
$$

for $0 \leq a<2^{n}$. These identities follow immediately from Theorem 3 and consideration of the binary expansion.

Finally, we observe that the sequences $\left(b_{i}\right)_{i \geq 0}$ and $\left(c_{i}\right)_{i \geq 0}$ are members of a much more general class of sequences, the so-called 2 -regular sequences [3]. In fact, even the sequence $\left(a_{i}\right)_{i \geq 0}$ is 2-regular, as our last theorem shows:

Theorem 6 We have

$$
\begin{aligned}
a_{4 i} & =a_{2 i} \\
a_{4 i+2} & =-a_{i}+2 a_{2 i}+2 a_{2 i+1}-a_{4 i+1} \\
a_{4 i+3} & =a_{2 i+1} \\
a_{8 i+1} & =a_{4 i+1} \\
a_{8 i+5} & =-a_{i}+2 a_{2 i}+3 a_{2 i+1}-a_{4 i+1}
\end{aligned}
$$

for all $i \geq 0$.

Proof. These relations follow easily from Theorem 4. For example, let us prove the identity for $a_{4 i+2}$. There are two cases to consider: when $i$ is even and when $i$ is odd.

If $i$ is even, say $i=2 k$, then

$$
\begin{aligned}
-a_{2 k}+2 a_{4 k}+2 a_{4 k+1}-a_{8 k+1} & =-F_{d_{2 k}+4}+2 F_{d_{4 k}+4}+2 F_{d_{4 k+1}+4}-F_{d_{8 k+1}+4} \\
& =-F_{d_{2 k}+4}+2 F_{d_{2 k}+4}+2 F_{d_{2 k}+5}-F_{d_{2 k}+5} \\
& =F_{d_{2 k}+4}+F_{d_{2 k}+5} \\
& =F_{d_{2 k}+6} \\
& =F_{d_{8 k+2}+4} \\
& =a_{8 k+2} .
\end{aligned}
$$

Here we have used the identities $d_{8 k+2}=d_{2 k}+2, d_{4 k}=d_{2 k}, d_{4 k+1}=d_{2 k}+1, d_{8 k+1}=d_{2 k}+1$, which are easily verified by considering the binary expansion of $k$.

If $i$ is odd, say $i=2 k+1$, then

$$
\begin{aligned}
-a_{2 k+1}+2 a_{4 k+2}+2 a_{4 k+3}-a_{8 k+5} & =-F_{d_{2 k+1}+4}+2 F_{d_{4 k+2}+4}+2 F_{d_{4 k+3}+4}-F_{d_{8 k+5}+4} \\
& =-F_{d_{2 k+1}+4}+2 F_{d_{2 k+1}+5}+2 F_{d_{2 k+1}+4}-F_{d_{2 k+1}+6} \\
& =F_{d_{2 k+1}+4}+F_{d_{2 k+1}+3} \\
& =F_{d_{2 k+1}+5} \\
& =F_{d_{8 k+6}+4} \\
& =a_{8 k+6} .
\end{aligned}
$$

Verification of the remaining identities is left to the reader.

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