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On lacunary formal power series and their continued fraction expansion

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To Andrzej Schinzel on his 60th birthday

Abstract. We note that the continued fraction expansion of a lacunary formal power series is a folded continued fraction with monomial partial quotients, and with the property that its convergents have denominators that are the sums of distinct monomials, that is, they are polynomials with coefficients 0, 1, and -1 only. Our results generalise, simplify and refine remarks of a previous note 'Convergents of folded continued fractions' (*Acta Arith.* LXXVII).

1. Introduction

We investigate the continued fraction expansion $[0, c_1(X)\ , c_2(X)\ , \dots]$ of series

$$f_{\mathcal{E},\Lambda}(X) = \sum_{h=0}^{\infty}(-1)^{\varepsilon_h}X^{-\lambda_h}$$

where $\Lambda = (\lambda_0, \lambda_1, \lambda_2, \dots)$ is some lacunary sequence of integers satisfying $\lambda_0 > 0$ and $\lambda_{h+1} > 2\lambda_h$ for $h = 0, 1, \dots$ and $\mathcal{E} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ is some infinite sequence of 0 's and 1 's.

The cases $\lambda_h = 2^h$ and $\lambda_h = 2^h - 1$ have been treated by Allouche *et alii* [1] and by van der Poorten and Shallit [3]. Notice that the former case is actually not included since we exclude equality $\lambda_{h+1} = 2\lambda_h$; however, $X^\nu \sum_{h=0}^{\infty}(-1)^{\varepsilon_h}X^{-2^h}$ does satisfy our conditions for integers $\nu > 0$. We eventually mention how our remarks do in fact inform also on the excluded case.

Recall that the regular continued fraction of a Laurent series in $\mathbb{F}((X^{-1}))$, $\mathbb{F}$ an arbitrary field, has partial quotients $c_1(X)$, $c_2(X)$, ... consisting of polynomials $c_h(X)$ of positive degree, whereas $c_0(X)$ is a polynomial that may be constant. Nonetheless, in the course of our argument we will entertain 'inadmissible' partial quotients not necessarily satisfying those conditions. Several lemmas provide simple rules for eliminating such improper partial quotients and yield our conclusions.

In the sequel we speak of the 'denominators q_h of the convergents of a continued fraction $[c_0\ , c_1\ , c_2\ , \dots]$', intending to refer to q_h in $[c_0\ , c_1\ , c_2\ , \dots\ , c_h] = p_h/q_h$.

It can be argued that the description does not well define the q_h. We here confirm that we intend the q_h to be determined by the matrix identity

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}.$$

Theorem. *Let* $[0\ ,\ c_1(X)\ ,\ c_2(X)\ ,\ \ldots]$ *be the continued fraction expansion of a lacunary power series* $f_{\mathcal{E},\Lambda}(X) = \sum_{h=0}^{\infty}(-1)^{\varepsilon_h}X^{-\lambda_h}$ *where* $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ *is some lacunary sequence of integers satisfying* $\lambda_0 > 0$ *and* $\lambda_{h+1} > 2\lambda_h$ *for* $h = 0,\ 1,\ \ldots$ *and* $\mathcal{E} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\ )$ *is some infinite sequence of* 0*'s and* 1*'s. Denote by* $p_h(X)/q_h(X)$ *the convergents of* $f_{\mathcal{E},\Lambda}(X)$*, to wit, the truncations* $[0\ ,\ c_1\ ,\ c_2\ ,\ \ldots\ ,\ c_h]$ *of that expansion. Then* (i) *the sequence* $c_1, c_2, c_3, \ldots$ *is a 'folded sequence': that is,* $c_{2^n(2h+1)} = (-1)^h c_{2^n}$ *for all nonnegative integers* n *and* h (*equivalently, always* $c_{2^h+n} = -c_{2^h-n}$ *for nonnegative integer* h *and integers* n *such that* $0 < n < 2^{h+1}$), *and its 'folds' are given by* $c_1 = (-1)^{\varepsilon_0}X^{\lambda_0}$ *and* $c_{2^{h+1}} = -(-1)^{\varepsilon_h}X^{\lambda_{h+1}-2\lambda_h}$ *for* $h = 0,\ 1,\ \ldots$; *and* (ii) *the denominators* q_h *of the convergents each are polynomials with coefficients* 0, 1 *or* -1 *only. More specifically,* q_h *is a sum of* u_h *distinct monomials each of the shape* $c_1c_2\cdots c_r$*, where* r *has the same parity modulo* 2 *as does* h*, and* (u_h) *is the so-called Stern-Brocot sequence defined by* $u_{2h+1} = u_h$*,* $u_{2h} = u_h + u_{h-1}$ *with initial conditions* $u_0 = 1$*,* $u_1 = 1$.

2. Proof of the Theorem

We rely on induction on the number of λ_h, showing that our presuming the theorem for all integers n and the exponents $\lambda_1,\ \lambda_2,\ \ldots\ ,\ \lambda_n$ entails what we wish to prove for the case of exponents $\lambda_0,\ \lambda_1,\ \lambda_2,\ \ldots\ ,\ \lambda_n$.

It seems barely worthy of remark that for just one exponent λ the expansion is $(-1)^{\varepsilon}X^{-\lambda} = [0\ ,\ (-1)^{\varepsilon}X^{\lambda}]$, so in that case

$$\begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (-1)^{\varepsilon}X^{\lambda} & 1 \end{pmatrix}.$$

Accordingly, we readily verify our allegations both that the continued fraction expansion is folded, here trivially, and that the q's are each a sum of distinct monomials of the appropriate shape; 1 is of course an empty product of consecutive partial quotients. Moreover, we easily count the number of monomials involved, and see their number is coherent with $u_0 = 1$ and $u_1 = 1$.

Now, set $\Lambda' = (\lambda_1, \lambda_2, \lambda_3, \ldots\ )$, define $\mathcal{E}'$ similarly, and denote the continued fraction expansion of the series $f_{\mathcal{E}',\Lambda'}(X)$ by $[0\ ,\ a_1\ ,\ a_2\ ,\ a_3\ ,\ a_4\ ,\ \ldots]$, where we suppose that $a_1 = (-1)^{\varepsilon_1}X^{\lambda_1}$ and $a_{2^h} = -(-1)^{\varepsilon_{h+1}}X^{\lambda_{h+1}-2\lambda_h}$ for $h > 0$. We further assume that the denominators q'_h of the convergents of this expansion are indeed each the sum of u_h distinct monomials each of the shape $a_1a_2\cdots a_r$, where r has the same parity modulo 2 as does h.

We start with the obvious

Lemma 1. (i) $\alpha = [0\,,-1\,,1\,,-1\,,0\,,-\alpha]$; (ii) $\alpha = [0\,,1\,,-1\,,1\,,0\,,-\alpha]$; (iii) $[\ldots\,,A\,,0\,,B\,,\ldots] = [\ldots\,,A+B\,,\ldots]$.

Lemma 2. $[a+1\,,b\,,c\,,\delta] = [a\,,1\,,-b\,,-1\,,c+1\,,\delta]$.

Proof. By Lemma 1,

$$\begin{aligned}[a+1\,,b\,,c\,,\delta] &= [a+1\,,0\,,-1\,,1\,,-1\,,0\,,-b\,,-c\,,-\delta]\\ &= [a\,,1\,,-b-1\,,-c\,,-\delta] = [a\,,1\,,-b-1\,,0\,,1\,,-1\,,1\,,0\,,c\,,\delta]\\ &= [a\,,1\,,-b\,,-1\,,c+1\,,\delta].\end{aligned}$$

Further, it is plain that

Lemma 3. *If* $\alpha = [a\,,b\,,c\,,d\,,e\,,f\,,g\,,\ldots]$, *then*

$$\alpha y = [ay\,,by^{-1}\,,cy\,,dy^{-1}\,,ey\,,fy^{-1}\,,gy\,,\ldots].$$

It is natural to have $\Lambda' - \lambda_0$ denote the sequence $(\lambda_1-\lambda_0, \lambda_2-\lambda_0, \lambda_3-\lambda_0, \ldots)$. Set $x = X^{\lambda_0}$ and write $a = a_1 = -a_3 = a_5 = \ldots$. Then our first induction assumption will have transformed to: $(-1)^{\varepsilon_0} + f_{\mathcal{E}',\Lambda'-\lambda_0}(X)$ has continued fraction expansion $[1\,,ax^{-1}\,,a_2x\,,-ax^{-1}\,,a_4x\,,ax^{-1}\,,\ldots]$. By Lemma 2 this expansion equals

$$[0\,,1\,,-ax^{-1}\,,-1\,,a_2x\,,1\,,ax^{-1}\,,-1\,,a_4x\,,1\,,-ax^{-1}\,,-1\,,\ldots].$$

Suppose we divide this continued fraction by x, thus obtaining the expansion of $(-1)^{\varepsilon_0}X^{-\lambda_0} + f_{\mathcal{E}',\Lambda'}(X) = f_{\mathcal{E},\Lambda}(X)$. By Lemma 3 we have

$$f_{\mathcal{E},\Lambda}(X) = [0\,,x\,,-ax^{-2}\,,-x\,,a_2\,,x\,,ax^{-2}\,,-x\,,a_4\,,x\,,-ax^{-2}\,,-x\,,\ldots],$$

confirming part (i) of the Theorem by induction.

Addition of ± 1 does not change the q''s — those of the 'old' expansion corresponding to $\mathcal{E}', \Lambda'$ — whilst the change from the sequence Λ' to $\Lambda' - \lambda_0$ entails that monomials $a_1a_2\cdots a_{2r-1}$ become $a_1a_2\cdots a_{2r-1}x^{-1}$ whilst monomials $a_1a_2\cdots a_{2r}$ stay unchanged. By our induction assumption we conclude that the q'_{2h+1} become $q'_{2h+1}x^{-1}$ and the q'_{2h} remain unchanged.

As particular cases of Lemma 2 we have $[a+1] = [a\,,1]$, more to the point $[a+1\,,b] = [a\,,1\,,-b\,,-1]$, $[a+1\,,b\,,c] = [a\,,1\,,-b\,,-1\,,c\,,1]$, and so on. Thus

$$\begin{aligned}[0\,,x] &= p_1/q_1 = p'_0/xq'_0,\\ [0\,,x\,,-ax^{-2}\,,-x] &= p_3/q_3 = p'_1/xq'_1x^{-1},\\ [0\,,x\,,-ax^{-2}\,,-x\,,a_2\,,x] &= p_5/q_5 = p'_2/xq'_2,\\ [0\,,x\,,-ax^{-2}\,,-x\,,a_2\,,x\,,ax^{-2}\,,-x] &= p_7/q_7 = p'_3/xq'_3x^{-1}\\ &\cdots\cdots.\end{aligned}$$

That is, we get the 'new' q's — those corresponding to $\mathcal{E}, \Lambda$ — by $q_{4h+1} = xq'_{2h}$, whilst $q_{4h-1} = q'_{2h-1}$. We note also that $c_{4h} = a_{2h}$ and $c_{4h-2} = (-1)^h a_{2h-1} x^{-2}$; whilst $c_{2h+1} = (-1)^h x$. Hence both

$$a_1 a_2 \cdots a_{2r-1} = (-1)^r c_2 c_4 \cdots c_{4r-2} x^{2r} = c_1 c_2 c_3 c_4 c_5 \cdots c_{4r-2} c_{4r-1}$$

and

$$x a_1 a_2 \cdots a_{2r} = (-1)^r c_2 c_4 \cdots c_{4r} x^{2r+1} = c_1 c_2 c_3 c_4 c_5 \cdots c_{4r-1} c_{4r+1}.$$

Thus the induction assumptions entail that q_{2h+1} is a sum of u_h terms of the appropriate form, confirming our claims at least for q_h with h odd.

However, of course $q_{2h+1} = (-1)^h x q_{2h} + q_{2h-1}$, so the 'missing' q's are given by $xq_{4h} = q_{4h+1} - q_{4h-1} = xq'_{2h} - q'_{2h-1}$ and $xq_{4h+2} = q_{4h+1} - q_{4h+3} = xq'_{2h} - q'_{2h+1}$ respectively. But xq'_{2h} is a sum of terms $xc_1c_2 \cdots c_{4r}$ whilst q'_{2h+1} is a sum of terms $c_1 c_2 \cdots c_{4r-1} = (-x) \cdot c_1 c_2 \cdots c_{4r-2}$. It follows that the $u_{2h} + u_{2h+1}$, respectively $u_{2h} + u_{2h-1}$, terms involved in the differences are distinct, and that all have a plus sign, proving our claims by induction.

3. Corollaries and details

Plainly the result we have just proved is purely formal. For example, the sequence of signs $\mathcal{E}$ plays no role of significance at all, and only appears in the definition of the 'folds' in the sequence of convergents. Not quite as trivially but somewhat similarly, the sequence of exponents Λ plays no role other than in defining the folds. Indeed, even its gap conditions, namely $\lambda_0 > 0$ and $\lambda_{h+1} > 2\lambda_h$, serve only to ensure that the partial quotients we cite are indeed admissible. Thus, for example, we could remark that the series

$$g_{\mathcal{E}}(X) = \sum_{h=0}^{\infty} (-1)^{\varepsilon_h} X^{-2^h}$$

has continued fraction expansion of the shape

$$[0 \,,\, X \,,\, a \,,\, -X \,,\, b \,,\, X \,,\, c \,,\, -X \,,\, d \,,\, X \,,\, e \,,\, -X \,,\, f \,,\, \ldots],$$

where a, b, c, … variously denote 1 or -1. Then invoking

Lemma 4. $[A \,,\, 1 \,,\, B, \gamma] = [A+1 \,,\, -B-1 \,,\, -\gamma]$
and $[A \,,\, -1 \,,\, B, \gamma] = [A-1 \,,\, -B+1 \,,\, -\gamma]$;

we fairly readily retrieve results of [1], indeed by the very equivalent of the method employed there.

Among various congenial rephrasings of the observation that the terms comprising the q_h are all of the shape $c_1 c_2 \cdots c_r$ is the remark that

The terms of the q_h belonging to $f_{\mathcal{E},\Lambda}$ are each of the shape

$$(-1)^{\nu(r,\mathcal{E})}X^{r(\Lambda)} = (-1)^{\nu(r,\mathcal{E})}X^{r_0\lambda_0+r_1(\lambda_1-\lambda_0)+r_2(\lambda_2-\lambda_1)+\ldots+r_s(\lambda_s-\lambda_{s-1})},$$

where $r = r_s2^s + r_{s-1}2^{s-1} + \ldots + r_12 + r_0$ displays r in base 2. It follows that the exponent $r(\Lambda)$ of X is an alternating sum

$$\lambda_s - \lambda_{s_1} + \lambda_{s_2} - \ldots + (-1)^k\lambda_{s_k},$$

where $s > s_1 > \ldots > s_k$. Then also the 'sign' $\nu(r,\mathcal{E})$ is given by

$$\delta(r) + \varepsilon_s + \varepsilon_{s_1} + \varepsilon_{s_2} + \ldots + \varepsilon_{s_k},$$

where $\delta(r)$ is precisely the number of $\overline{1}$'s in the first r terms of the paperfolding sequence $\underline{\underline{1}}, \underline{\underline{\overline{1}}}, \overline{1}, \underline{\underline{\overline{1}}}, 1, 1, \overline{1}, \underline{\underline{\overline{1}}}, 1, \overline{1}, \overline{1}, 1, 1, 1, \overline{1}, \underline{\underline{\overline{1}}}, 1, \overline{1}, \overline{1}, \overline{1}, 1, 1, \overline{1}, 1, 1, \overline{1}, \ldots$, all of whose unfolding instructions, bar the first, are $\overline{1} := -1$. In fact, $\delta(r) = \sum r_{i+1}(1-r_i)$ is the number of 10's in the binary expansion $r_sr_{s-1}\ldots r_0$ of r.

This and more general such counts may be found in [2].

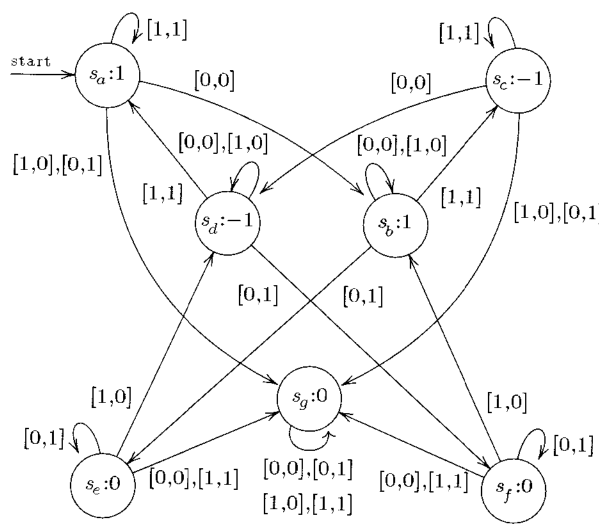
The remaining question is just which r provide the terms of q_h. However, the reader will have observed that the results of [1] apply here. Namely, $(-1)^{\delta(r)}X^r$ occurs as a term in the denominator of the h-th convergent of $X\sum_{h\geq 0}X^{-2^h}$ if and only if $(-1)^{\nu(r,\mathcal{E})}X^{r(\Lambda)}$ is a term of q_h in our generalisation. Unfortunately, the discussion in [1] is marred by a confusing misTEX. It suffices to remark that, to discover which r occur, it is enough to study the matter in characteristic 2 when the continued fraction expansion is just $[1\ ,\ \overline{X}\,] = 1 + \frac{1}{2}X(-1+\sqrt{1+4X^{-2}})$. Plainly, the denominators t_h of its convergents satisfy $t_0 = 1$, $t_1 = X$, and $t_{h+2} = Xt_{h+1}+t_h$ for integers $t \geq 0$, so they are given by the generating function $\sum t_hz^h = 1/(1-Xz-z^2)$. This is $t_h = \sum_{i\geq 0}\binom{h-i}{i}X^{h-2i}$. That is, only $r = h-2i$ can possibly occur, in accordance with our claims about the parity of r relative to h. Finally, $r = h-2i$ does in fact occur only if the binomial coefficient is odd, which is exactly when adding $h-2i$ to i in base 2 involves no 'carries'.

The automaton below tells this and more. Namely, with initial state s_a, one inputs pairs (h,r), of course displayed in binary, and starting with their least significant digits. The final state maps to 0 if r does not provide a term of q_h; otherwise it maps to $(-1)^{\delta(r)}$. The transition table, together with the mapping $s_a \mapsto 1$, $s_b \mapsto 1$, $s_c \mapsto 0$, $s_d \mapsto -1$, $s_e \mapsto 0$, $s_f \mapsto 0$, $s_g \mapsto -1$, conveys the same information as does the automaton. The automaton was drawn using XY-pic, by Ross Moore, one of its authors.*

For example, $r = 5$ provides a term of q_9, and $\delta(5) \equiv 1 \pmod 2$. To see that the automaton reports this, input the pair $(1001, 0101)$. The symbol $[1,1]$ sends state s_a to itself. Then $[0,0]$ send s_a to s_b. Next, $[0,1]$ takes s_b to s_c. Last, $[1,0]$ sends s_c to s_d, which indeed maps to -1. Notice that prepending an extra $[0,0]$ does not change the value to which the final state maps, as of course it mustn't.

* See <http://www-math.mpce.mq.edu.au/~ross/Xy-pic.html>

	[0, 0]	[1, 0]	[0, 1]	[1, 1]
s_a	s_b	s_g	s_g	s_a
s_b	s_b	s_b	s_c	s_c
s_c	s_d	s_c	s_c	s_c
s_d	s_d	s_d	s_f	s_a
s_e	s_g	s_d	s_e	s_g
s_f	s_g	s_b	s_f	s_g
s_g	s_g	s_g	s_g	s_g



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References

[1] Allouche, J.-P., Lubiw, A., Mendès France, M., van der Poorten, A.J., Shallit, J., Convergents of folded continued fractions. Acta Arith. 77 (1996), 77–96.

[2] Mendès France, M., The inhomogeneous Ising chain and paperfolding. In: Number theory and physics (ed. by J.M. Luck, P. Moussa, and M. Waldschmidt; Springer Proc. Phys. 47), 195–202. Springer, Berlin 1990.

[3] van der Poorten A.J., Shallit, J.O., Folded continued fractions. J. Number Theory 40 (1992), 237–250.