Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients

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ABSTRACT

Let θ be a real number with continued fraction expansion $\theta = [a_0, a_1, a_2, \ldots]$, and let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with integer entries and with $|\det(M)| \neq 0$. If θ has bounded partial quotients, then $\frac{a\theta+b}{c\theta+d} = [a_0^*, a_1^*, a_2^*, \ldots]$ also has bounded partial quotients. More precisely, if $a_j \leq K$ for all sufficiently large j, then $a_j^* \leq |\det(M)|(K+2)$ for all sufficiently large j. We also give a weaker bound valid for all a_j^* with $j \geq 1$.

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1. Introduction

Let θ be a real number whose expansion as a simple continued fraction is

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

and set

$$K(\theta) := \sup_{i \geq 1} \; a_i \; , \tag{1.1}$$

where we adopt the convention that $K(\theta) = +\infty$ if θ is rational. We say that θ has *bounded partial quotients* if $K(\theta)$ is finite. We also set

$$K_\infty(\theta) := \limsup_{i \geq 1} \; a_i \; , \tag{1.2}$$

where $K_\infty(\theta) = +\infty$ if θ is rational. Certainly $K_\infty(\theta) \leq K(\theta)$, and $K_\infty(\theta)$ is finite if and only if $K(\theta)$ is finite. A survey of results about real numbers with bounded partial quotients is given in [16].

The property of having bounded partial quotients is equivalent to θ being a *badly approximable number*, which is that

$$\liminf_{q \to \infty} q||q\theta|| > 0 \; ,$$

in which $||x|| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$ denotes the distance from x to the nearest integer.

This note proves two quantitative versions of the "folk theorem" that if θ has bounded partial quotients and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then $\psi = \frac{a\theta+b}{c\theta+d}$ also has bounded partial quotients.

The first result bounds $K_\infty(\frac{a\theta+b}{c\theta+d})$ in terms of $K_\infty(\theta)$ and depends only on $|\det(M)|$.

Theorem 1.1. *Let θ have a bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $|\det(M)| \neq 0$, then*

$$K_\infty\left(\frac{a\theta+b}{c\theta+d}\right) \leq |\det(M)|(K_\infty(\theta)+2)\ . \tag{1.3}$$

The second result bounds $K(\frac{a\theta+b}{c\theta+d})$ in terms of $K(\theta)$, and depends on the entries of M.

Theorem 1.2. *Let θ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $|\det(M)| \neq 0$, then*

$$K\left(\frac{a\theta+b}{c\theta+d}\right) \leq |\det(M)|(K(\theta)+2) + |c(c\theta+d)|\ . \tag{1.4}$$

The last term in (1.4) can be bounded in terms of the partial quotient a_0 of θ, since

$$|c\theta + d| \leq |c|(|a_0|+1) + |d| \leq |ca_0| + |c| + |d|\ .$$

Theorem 1.2 gives no bound for the partial quotient $A_0 := \lfloor \frac{a\theta+b}{c\theta+d} \rfloor$ of $\frac{a\theta+b}{c\theta+d}$.

Chowla [2] proved in 1931 that $K(\frac{a}{c}\theta) < 2ac(K(\theta)+1)^3$, a result rather weaker than Theorem 1.2.

We obtain Theorem 1.1 and Theorem 1.2 from stronger bounds that relate the Diophantine approximation constants of θ and $\frac{a\theta+b}{c\theta+d}$, which appear below as Theorem 3.2 and Theorem 4.1, respectively. Theorem 3.2 is a simple consequence of a result of Cusick and Mendès France [4] concerning the Lagrange constant of θ.

The continued fraction of $\frac{a\theta+b}{c\theta+d}$ can be directly computed from that for θ, as was observed in 1894 by Hurwitz [8], who gave an explicit formula for the continued fraction of 2θ in terms of that of θ. In 1947 Hall [6] gave a method to compute the continued fraction for general $\frac{a\theta+b}{c\theta+d}$. Let $\mathcal{M}(n,\mathbb{Z})$ denote the set of $n \times n$ integer matrices. Raney [14] gave for each $M =$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{Z})$ with $\det(M) \neq 0$ an explicit finite automaton to compute the additive continued fraction of $\frac{a\theta+b}{c\theta+d}$ from the additive continued fraction of θ.

In connection with the bound of Theorem 1.1, Davenport [5] observed that for each irrational θ and prime p there exists some integer $0 \leq a < p$ such that $\theta' = \theta + \frac{a}{p}$ has infinitely many partial quotients $a_n(\theta') \geq p$. Mendès France [12] then showed that there exists some $\theta' = \theta + \frac{a}{p}$ having the property that a positive portion of the partial quotients $a_n(\theta')$ of θ' are $\geq p$.

Some other related results appear in Mendès France [10, 11]. Basic facts on continued fractions appear in [1, 7, 9, 17].

2. Badly Approximable Numbers

Recall that the continued fraction expansion of an irrational real number $\theta = [a_0, a_1, \ldots]$ is determined by

$$\theta = a_0 + \theta_0 \ , \quad 0 < \theta_0 < 1 \ ,$$

and for $n \geq 1$ by the recursion

$$\frac{1}{\theta_{n-1}} = a_n + \theta_n \ , \quad 0 < \theta_n < 1 \ .$$

The *n-th complete quotient* α_n of θ is

$$\alpha_n := \frac{1}{\theta_n} = [a_j, a_{j+1}, a_{j+2}, \ldots] \ .$$

The *n-th convergent* $\frac{p_n}{q_n}$ of θ is

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] \ ,$$

whose denominator is given by the recursion $q_{-1} = 0, q_0 = 1$, and $q_{n+1} = a_{n+1} q_n + q_{n-1}$. It is well known (see [7, §10.7]) that

$$||q_n\theta|| = |q_n\theta - p_n| = \frac{1}{q_n\alpha_{n+1} + q_{n-1}} \ . \tag{2.1}$$

Since $a_{n+1} \leq \alpha_{n+1} < a_{n+1} + 1$ and $q_{n-1} \leq q_n$, this implies that

$$\frac{1}{a_{n+1} + 2} < q_n ||q_n\theta|| \leq \frac{1}{a_{n+1}} \ , \tag{2.2}$$

for $n \geq 0$.

For an irrational number θ define its *type* $L(\theta)$ by

$$L(\theta) = \sup_{q \geq 1} \ (q||q\theta||)^{-1} \ ,$$

and define the *Lagrange constant* $L_\infty(\theta)$ of θ by

$$L_\infty(\theta) = \limsup_{q \geq 1} \ (q||q\theta||)^{-1} \ .$$

Again we use the convention that $L(\theta) = L_\infty(\theta) = +\infty$ if θ is rational.

The best approximation properties of continued fraction convergents give

$$L(\theta) = \sup_{n \geq 0} \ (q_n||q_n\theta||)^{-1} \tag{2.3}$$

and

$$L_\infty(\theta) = \limsup_{n \geq 0} \ (q_n||q_n\theta||)^{-1} \ . \tag{2.4}$$

There are simple relations between these quantities and the partial quotient bounds $K(\theta)$ and $K_\infty(\theta)$, cf. [15, pp. 22–23].

Lemma 2.1. *For any irrational θ with bounded partial quotients, we have*

$$K(\theta) \leq L(\theta) \leq K(\theta) + 2 \ . \tag{2.5}$$

Proof. This is immediate from (2.2) and (2.3). $\square$

Lemma 2.2. *For any irrational θ with bounded partial quotients*

$$K_\infty(\theta) \leq L_\infty(\theta) \leq K_\infty(\theta) + 2 \ . \tag{2.6}$$

Proof. This is immediate from (2.2) and (2.4). $\square$

Although we do not use it in the sequel, we note that both inequalities in (2.6) can be slightly improved. Since $q_n \leq (a_n + 1)q_{n-1}$, (2.1) yields

$$q_n||q_n\theta|| \leq \frac{1}{\alpha_{n+1} + \frac{q_{n-1}}{q_n}} \leq \frac{1}{a_{n+1} + 1/(a_n + 1)} \ .$$

Since $a_n \leq K_\infty(\theta)$ from some point on, this and (2.4) yield

$$L_\infty(\theta) \geq K_\infty(\theta) + \frac{1}{K_\infty(\theta) + 1} \ . \tag{2.7}$$

Next, from (2.1) we have

$$\begin{aligned} q_n||q_n\theta|| &= \frac{q_n}{\alpha_{n+1}q_n + q_{n-1}} \\ &= \frac{1}{a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}}. \end{aligned}$$

Hence

$$(q_n||q_n\theta||)^{-1} = a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}.$$

Let $K = K_\infty(\theta)$. Then for all n sufficiently large we have

$$\alpha_{n+2} \geq 1 + \frac{1}{K+1} = \frac{K+2}{K+1},$$

so

$$\begin{aligned} (q_n||q_n\theta||)^{-1} &\leq K + \tfrac{K+1}{K+2} + 1 \\ &= K + 2 - \tfrac{1}{K+2}. \end{aligned}$$

We conclude that

$$L_\infty(\theta) \leq K_\infty(\theta) + 2 - \frac{1}{K_\infty(\theta) + 2} \ . \tag{2.8}$$

3. Lagrange Spectrum and Proof of Theorem 1.1.

The Lagrange constant satisfies $L_\infty(\theta) \geq \sqrt{5}$ for all θ, and is also given by the formula

$$L(\theta) = \limsup_{j\to\infty}([a_j, a_{j+1}, \ldots] + [0, a_{j-1}, a_{j-2}, \ldots, a_1]) \ ; \tag{3.1}$$

see Cusick and Flahive [3].

Given an integer matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(M) \neq 0$, set

$$M(\theta) := \frac{a\theta + b}{c\theta + d} \ , \tag{3.2}$$

and note that $M_1(M_2(\theta)) = M_1 M_2(\theta)$.

Lemma 3.1. *If M is an integer matrix with* $\det(M) = \pm 1$, *then*

$$L_\infty(M(\theta)) = L_\infty(\theta) \ .$$

Proof. This is well-known, cf. [13] and [4, Lemma 1], and is deducible from (3.1). □

The main result of Cusick and Mendès France [4] yields:

Theorem 3.2. *For any integer* $m \geq 1$, *let*

$$G_m = \{M \in \mathcal{M}(2,\mathbb{Z}) : |\det(M)| = m\} \ .$$

Then for any irrational number θ,

$$\sup_{M\in G_m} (L_\infty(M(\theta))) = mL(\theta) \ . \tag{3.3}$$

Proof. Theorem 1 of [4] states that

$$\max_{\substack{a,b,d \\ ad=m \\ 0\leq b<d}} \left(L_\infty\left(\frac{a\theta+b}{d}\right)\right) = mL(\theta) \ . \tag{3.4}$$

Let $GL(2,\mathbb{Z})$ denote the group of 2×2 integer matrices with determinant ± 1. We need only observe that for any M in G_m there exists some $\tilde{M} \in GL(2,\mathbb{Z})$ such that $\tilde{M}M = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $a'd' = m$ and $0 \leq b' < d'$. For if so, and $\psi = \frac{a\theta+b}{c\theta+d}$, then Lemma 3.1 gives

$$L_\infty(\psi) = L_\infty(\tilde{M}(\psi)) = L_\infty(\tilde{M}M(\theta)) = L_\infty\left(\frac{a'\theta+b'}{d'}\right) \ ,$$

whence (3.4) implies (3.3). Finally set $\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and we need

$$Ca + Dc = 0 \ .$$

Take $C = \frac{\operatorname{lcm}(a,c)}{a}$ and $D = -\frac{\operatorname{lcm}(a,c)}{c}$. Then $\gcd(C,D) = 1$, so we may complete this row to a matrix $\tilde{\tilde{M}} \in GL(2,\mathbb{Z})$. Multiplying this by a suitable matrix $\begin{bmatrix} \pm 1 & c \\ 0 & \pm 1 \end{bmatrix}$ yields the desired $\tilde{M}$. □

Proof of Theorem 1.1. Theorem 3.2 gives $L_\infty(M(\theta)) \leq \det(M)L(\theta)$. Now apply Lemma 2.2 twice to get

$$\begin{aligned} K_\infty(M(\theta)) &\leq L_\infty(M(\theta)) \\ &\leq |\det(M)|L_\infty(\theta) \\ &\leq |\det(M)|(K_\infty(\theta)+2) \ . \quad \square \end{aligned}$$

4. Numbers of Bounded Type and Proof of Theorem 1.2

Recall that the *type* $L(\theta)$ of θ is the smallest real number such that $q||q\theta|| \geq \frac{1}{L(\theta)}$ for all $q \geq 1$.

Theorem 4.1. *Let θ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with* $\det(M) \neq 0$, *then*

$$L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)|L(\theta) + |c(c\theta + d)| \ . \tag{4.1}$$

Proof. Set $\psi = \frac{a\theta+b}{c\theta+d}$. Suppose first that $c = 0$ so that $|\det(M)| = |ad| > 0$. Then $L(\psi) \geq \frac{1}{x}$, where

$$x := q||q\psi|| = q||q\left(\frac{a\theta + b}{d}\right)|| = q|q\left(\frac{a\theta + b}{d}\right) - p| \ . \tag{4.2}$$

We have

$$\begin{aligned} |ad|x &= |aq|\ |aq\theta + (bq - dp)| \\ &\geq |aq|\ ||aq\theta|| \geq \frac{1}{L(\theta)} \ . \end{aligned} \tag{4.3}$$

For any $\epsilon > 0$ we may choose q in (4.2) so that $\frac{1}{x} \geq L(\psi) - \epsilon$. Then

$$|\det(M)|L(\theta) = |ad|L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon \ . \tag{4.4}$$

Letting $\epsilon \to 0$ yields (4.1) when $c = 0$.

Suppose now that $c \neq 0$. Again $L(\psi) \geq \frac{1}{x}$ where

$$x := q||q\psi|| = q|q\left(\frac{a\theta + b}{c\theta + d}\right) - p| \ .$$

We have

$$|c\theta + d|x = q|(qa - pc)\theta - (pd - qb)| \ , \tag{4.5}$$

so that

$$\begin{aligned} |c\theta + d|\left|\frac{qa - pc}{q}\right|x &= |qa - pc|\ |(qa - pc)\theta - (pd - qb)| \\ &\geq |qa - pc|\ ||(qa - pc)\theta|| \ . \end{aligned} \tag{4.6}$$

We first treat the case $qa - pc = 0$. Now

$$\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} qa - pc \\ pd - qb \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

since $\det \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \det(M) \neq 0$. Thus if $qa - pc = 0$ then $|pd - qb| \geq 1$, hence (4.5) gives

$$|c\theta + d|x = q|pd - qb| \geq 1 . \tag{4.7}$$

It follows that $qa - pc \neq 0$ provided that

$$\frac{1}{x} > |c\theta + d| . \tag{4.8}$$

We next treat the case when $qa - pc \neq 0$. Now from the definition of $L(\theta)$ we see

$$|qa - pc| \; ||(qa - pc)\theta|| \geq \frac{1}{L(\theta)} . \tag{4.9}$$

Given $\epsilon > 0$, we may choose q so that $\frac{1}{x} \geq L(\psi) - \epsilon$, and we obtain from (4.6) and (4.9) that

$$|c\theta + d| \; |\frac{qa - pc}{q}|L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon . \tag{4.10}$$

However, the bound

$$|q\left(\frac{a\theta + b}{c\theta + d}\right) - p| \leq \frac{1}{2}$$

implies that

$$\begin{aligned} |q\left(\frac{a}{c}\right) - p| &\leq |q\left(\frac{a\theta + b}{c\theta + d}\right) - q\left(\frac{a}{c}\right)| + \frac{1}{2} \\ &\leq q|\det(M)| \; |\frac{1}{c(c\theta + d)}| + \frac{1}{2} . \end{aligned}$$

Multiplying this by $\frac{c}{q}$ and substituting with (4.10) yields

$$L\left(\frac{a\theta + b}{c\theta + d}\right) - \epsilon \leq |\det(M)|L(\theta) + \frac{1}{2}\frac{|c(c\theta + d)|}{q} . \tag{4.11}$$

Letting $\epsilon \to 0$ and using $q \geq 1$ yields

$$L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det(M)|L(\theta) + \frac{1}{2}|c(c\theta + d)| , \tag{4.12}$$

provided that (4.8) holds. Now (4.8) fails to hold only if

$$L\left(\frac{a\theta + b}{c\theta + d}\right) \leq |c\theta + d| . \tag{4.13}$$

The last two inequalities imply (4.1) when $c \neq 0$. $\square$

Proof of Theorem 1.2. Applying Theorem 4.1 and Lemma 2.1 gives

$$\begin{aligned} K\left(\frac{a\theta+b}{c\theta+d}\right) &\leq L\left(\frac{a\theta+b}{c\theta+d}\right) \\ &\leq |\det(M)|L(\theta) + |c(c\theta+d)| \\ &\leq |\det(M)|(K(\theta)+2) + |c(c\theta+d)| \ , \end{aligned}$$

which is the desired bound. $\square$

Remark. The proof method of Theorem 4.1 can also be used to directly prove the upper bound

$$L_\infty(M(\theta)) \leq |\det(M)|L_\infty(\theta) \tag{4.14}$$

in Theorem 3.1, from which Theorem 1.1 can be easily deduced. We sketch a proof of (4.14) for the case $\psi = \frac{a\theta+b}{c\theta+d}$ with $c \neq 0$. For any $\epsilon^* > 0$ and all sufficiently large $q^* \geq q^*(\epsilon^*)$, we have

$$q^*||q^*\theta|| \geq \frac{1}{L_\infty(\theta)+\epsilon^*} \ .$$

We choose $q = q_n(\psi)$ for sufficiently large n, and note that

$$q^* = |q_n(\psi)a - p_n(\psi)c| \to \infty$$

as $n \to \infty$, since ψ is irrational. We can then replace (4.9) by

$$q^*||q^*\theta|| \geq \frac{1}{L_\infty(\theta)+\epsilon^*} \ .$$

This yields (4.12) with $L(\theta)$ replaced by $L_\infty(\theta)+\epsilon^*$, and letting $q\to\infty$, $\epsilon\to 0$ and $\epsilon^*\to 0$ yields (4.13).

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