THE PRIME FACTORIZATION OF 1

1. Introduction. The answer to the question "Is one a prime?" is hardly one of earth-shaking consequences. Nevertheless, there has been far from complete argreement among mathematicians on the answer.

It is generally accepted today that 1 is a number that is neither prime nor composite. One important exception occurs in the table of prime numbers by D. N. Lehmer [1] . Here, 1 is given as a prime. Another exception occurs in [2, p. 211] where it is stated that "from the humble 2, the only even prime, and 1, the smallest of the odd primes, [the prime numbers] rise in an unending succesion aloof and irrefrangible."

Since it is impossible to "prove" whether or not 1 is a prime, here we will obtain n intuitive definition for the prime factorization of 1, and then show how well the definition holds together.

2. A Factoring Program. For our discussion, we will need a program to separate integers greater than 1 into their prime factors. The particular definition does not matter; for example, the one in [3, p. 10] can be used.

Observe the following examples:

```
  FACTOR 2
2
      FACTOR 8
2 2 2
      FACTOR 30
2 3 5
      FACTOR 45
3 3 5
      ⍴FACTOR 2
1
```

The question arises of how to define the result of *FACTOR* 1 . If 1 is a prime, then the result should obviously be 1. However, the following identity then would not hold for *A* equal to 1:

```
⍴FACTOR A×B ↔ (⍴FACTOR A) + ⍴FACTOR B
```

Letting *A* equal 1, we find that

```
⍴FACTOR B ↔ (⍴FACTOR 1) + ⍴FACTOR B
        0 ↔ ⍴FACTOR 1
       ⍳0 ↔ FACTOR 1
```

Since (⍴*FACTOR X*) is the number of prime factors of *X* , we are forced to conclude that 1 is a number with 0 prime factors. In other words, the prime factorization of 1 is the empty vector.

3. Is our definition consistent? Here we give several examples of the value of the definition of *FACTOR* 1 as ⍳0 .

First we note the following identity

```
X ↔ ×/FACTOR X
```

For *X* equal to 1, we have 1 ↔ ×/⍳0 , and the definition holds.

Second, we introduce the function *ANALYZE* with the following definition:

```
     ∇ Z←ANALYZE X;R
[1]  R←((X⍳X)=⍳⍴X)/X
[2]  2←R, [1.5] +/R∘.=X
   ∇
```

The purpose of *ANALYZE* is to form a matrix of dimension (*K*,2), where *K* is the number of distinct elements in the vector argument *X* . (*ANALYZE X*)[;1] gives the distinct elements in *X* , and *ANALYZE X*)[;2] gives the number of occurrences of each distinct element.

If *R*←*ANALYZE FACTOR X* , then we have the following identity

```
X ↔ R[;1]×.*R[;2]
```

If *X* is 1, then (ρ*R*) ↔ 0 2 and 1 ↔ (ι0)×.*ι0 . Again, the definition of *FACTOR* 1 as ι0 holds.

Third, we introduce the function *MOEBIUS* which is the number-theoretic function u (*X*) :

```
    ∇ Z←MOEBIUS X;R
[1] R←(ANALYZE_FACTOR Z)[;2]
[2] Z←(≥/1=R)×¯1*+/R
  ∇
```

In most number theory texts, for example, in [4] , the following definition for u (×) , is given:

(A) u(1) = 1, by definition.

(B) u(x) = 0 if x is divisible by a square > 1.

(C) u(x) = +1 if x is square-free and contains an even number of prime factors.

(D) u(x) = -1 if x is square-free and contains an odd number of prime factors.

Note that u(1) is <u>defined</u> as 1 in a special case. However, if use our definition of 1 as a number with 0 prime factors (hence, even number of prime factors), then part (A) of the definition of u(x) is unnecessary--it is replaced with part (C).

Indeed,

```
      MOEBIUS 1
1
```

Fourth, we introduce the function *TOTIENT* , which is the number-theoretic function ϕ(×) . This function is traditionally defined as follows:

ϕ(1) = 1, by definition.

ϕ(×) = the number of positive integers < × and relatively prime to x.

Again, note that ϕ(1) is <u>defined</u> as 1 in a special case.

Here is the *APL* definition of the function *TOTIENT* :

```
    ∇ Z←TOTIENT X
[1]  Z←X××/1-÷(ANALYZE FACTOR X)[;1]
  ∇
```

However, with our definition of the prime factorization of 1 as ι0 we have

```
      TOTIENT 1
1
```

Finally, we observe that the positive integers can be split into three classes:

(1) The numbers such that (ρ*FACTOR X*) > 1. These are the compositve numbers 4, 6, 8, 9, 10, 12, 14, 15,...

(2) The numbers such that (ρ*FACTOR X*) = 1. These are the prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, ...

(3) The numbers such that (ρ*FACTOR X*) = 0. The only number with (ρ*FACTOR X*) equal to 0 is 1. This explains why 1 is considered neither prime nor composite.

<u>References</u>

1. D. N. Lehmer, List of Prime Numbers from 1 to 10,006,721, Carnegie Institute, Washington, Pub. 165 (1914).

2. Albert H. Beiler, Recreations in the Theory of Numbers, Dover, 1964.

3. *SHARE*APL*/360 Newsletter, Number 2, July, 1969.

4. J. V. Uspensky and M. A. Heaslet. Elementary Number Theory, McGraw- Hill, 1939.

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