Infinite Products Associated with Counting Blocks in Binary Strings

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Abstract.

Let w be a string of 0's and 1's, and let $a_w(n)$ be the function which counts the number of (possibly overlapping) occurrences of w in the binary expansion of n. We show that there exists an effectively computable rational function $b_w(n)$ such that

$$\sum_{n\geq 0} \log_2(b_w(n)) X^{a_w(n)} = -\frac{1}{1-X}.$$

By setting $X = -1$ and exponentiating, we recover previous results and also obtain some new ones; for example,

$$\prod_{n\geq 1} \left(\frac{2n}{2n+1}\right)^{(-1)^{a_0(n)}} = \frac{\sqrt{2}}{2}.$$

Our work is a generalization of previous results of D. Woods, D. Robbins, H. Cohen, M. Mendès France, and the authors.

I. Introduction.

Let $s_q(n)$ denote the sum of the digits of the nonnegative integer n when written in base q. Woods and Robbins [9] [7] showed that

$$\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{s_2(n)}} = \frac{\sqrt{2}}{2} \tag{1}$$

This formula was generalized by the second author to bases other than 2 using methods of real analysis [8]. Later, the first author and H. Cohen found more general results using Dirichlet series [1].

In a joint paper with Cohen and M. Mendès France [3] the authors showed that

$$\sum_{n\geq 0} X^{s_q(n)} \log_q\left(\frac{n+1}{q\lfloor\frac{n}{q}\rfloor+q}\right) = -\frac{1}{1-X} \tag{2}$$

for X in a suitable region of convergence, from which the results of Woods and Robbins easily follow upon letting $X=-1$ and $q=2$.

In [3] it was also shown that

$$\sum_{n\geq 0} X^{u(n)} \log_2\left(\frac{(2n+1)^2}{(n+1)(4n+1)}\right) = -\frac{1}{1-X},$$

where $u(n)$ is the Rudin-Shapiro function, which counts the number of occurrences of '11' in the binary expansion of n. This formula naturally suggests the existence of similar formulas corresponding to the counting of *any* binary string.

The existence of such formulas is the question we address in this paper. For any finite nonempty block w of 0's and 1's, we define $a_w(n)$ as the number of occurrences of w in the binary expansion of n. With this quantity we associate an infinite series of the form

$$\sum_n \log_2(b_w(n))X^{a_w(n)}$$

whose sum is also $-\frac{1}{1-X}$. This allows us to evaluate some novel infinite products, as well as recover previous results.

II. Notation.

Let w be a string or block of 0's and 1's (i. e. $w\in(0+1)^*$). Let $v:(0+1)^*\to\mathbb{N}$ be the map that assigns to w its value when interpreted in base 2; e. g. $v(0101)=5$. Let $|w|$ denote the *length* of w, i. e. the number of symbols in the string w.

Let w be nonempty and let $a_w(n)$ count the number of (possibly overlapping) occurrences of the block w in the binary expansion of n. For example, $a_{11}(15)=3$. Some clarification is needed in the case where w starts with a 0; if $w\neq 0^j$, then in evaluating

$a_w(n)$ we assume that the binary expansion of n starts with an arbitrarily long prefix of 0's. Thus $a_{010}(5) = 1$, since we consider the binary expansion of 5 to be $00\ldots 00101$. This definition of $a_w(n)$ is appropriate for all cases except $w = 0^j$; in this case we use the binary expansion of n which starts with a 1. Thus $a_{00}(4) = 1$.

If w and z are strings, let $a'_w(z)$ count the number of (possibly overlapping) occurrences of w in z. Note that here the string z is *not* extended with leading zeroes.

Finally, we define

$$L(x) = \log_2\left(\frac{x}{x+1}\right).$$

III. Some infinite series.

Our goal is to prove the result mentioned at the end of the introduction. First, however, we will prove the following

Lemma 1.

Let w be a nonempty string of 0's and 1's and let $a_w(n)$ be as defined above. Let g and h be integers. Then for all $k \geq 0$, the series

$$\sum_{\substack{n\geq 1\\ a_w(gn+h)=k}} \frac{1}{n}$$

converges.

Proof.

It suffices to prove that

$$\sum_{\substack{n\geq 1\\ a_w(gn+h)=k}} \frac{1}{gn+h}$$

converges. But this is majorized by

$$\sum_{\substack{n\geq 1\\ a_w(n)=k}} \frac{1}{n},$$

so it suffices to prove that this last series converges.

Let $b = 2^{|w|}$, let N be a positive integer, and let l be defined by $b^{l-1} \leq N \leq b^l - 1$, so that $l \leq 1 + \frac{\log N}{\log b}$. We may suppose $l \geq k$. Let us write $s_w(n)$ for the function which counts the number of occurrences of the "digit" w when n is written in base b. Then

$$S(N) = \sum_{\substack{0\leq n\leq N\\ a_w(n)=k}} 1 \leq \sum_{\substack{0\leq n\leq b^l-1\\ s_w(n)\leq k}} 1 = \sum_{0\leq j\leq k} \binom{l}{j}(b-1)^{l-j}.$$

Now bound $(b-1)^{l-j}$ by $(b-1)^l$ and $\binom{l}{j}$ by $\frac{l^j}{j!} \le l^k$. Thus

$$S(N) = \sum_{\substack{0 \le n \le N \\ a_w(n)=k}} 1 \le (k+1) l^k (b-1)^l \le c_k (\log N)^k N^{\frac{\log(b-1)}{\log b}}.$$

Now put

$$a(n) = \begin{cases} 1 & \text{if } a_w(n) = k, \\ 0 & \text{otherwise.} \end{cases}$$

We see that

$$S(N) = \sum_{\substack{0 \le n \le N \\ a_w(n)=k}} 1 = \sum_{0 \le n \le N} a(n),$$

from which we get

$$\sum_{\substack{1 \le n \le N \\ a_w(n)=k}} \frac{1}{n} = \sum_{1 \le n \le N} \frac{a(n)}{n} = \sum_{1 \le n \le N} \frac{S(n) - S(n-1)}{n}$$
$$= -S(0) + \frac{S(N)}{N} + \sum_{1 \le n \le N-1} \frac{S(n)}{n(n+1)}$$

and, using the bound determined above for $S(n)$, this quantity tends to a finite limit as $N \to \infty$. ■

Theorem 2.

Let w be a nonempty string of 0's and 1's and

$$g = 2^{|w|-1}, \quad h = \left\lfloor \frac{v(w)}{2} \right\rfloor.$$

Then, for all $k \ge 0$,

$$\sum_{\substack{n \\ a_w(gn+h)=k}} L(2gn + v(w)) = -1, \tag{3}$$

where the sum is over $n \ge 1$ if $w = 0^j$ and $n \ge 0$ otherwise.

Proof.

Note that we claim that the sum is -1, independent of k. Since

$$L(2gn + v(w)) = \frac{-1}{2(\log 2) gn} + O\left(\frac{1}{n^2}\right),$$

Lemma 1 ensures convergence of these series.

The proof is divided into three cases: (I) w ends in a 1; (II) w ends in a 0 but $w \neq 0^j$; and (III) $w = 0^j$.

Case I: w ends in a 1.

Let $d_w(k)$ be defined by

$$d_w(k) = \sum_{\substack{n \geq 0 \\ a_w(n)=k}} L(2n+1).$$

By writing $n = gr + m$, with $r \geq 0$ and $0 \leq m \leq g-1$, we see that

$$d_w(k) = \sum_{m=0}^{g-1} \sum_{\substack{r \geq 0 \\ a_w(gr+m)=k}} L(2gr+2m+1).$$

Similarly, if we let

$$e_w(k) = \sum_{\substack{n \geq 0 \\ a_w(2n+1)=k}} L(2n+1),$$

then

$$e_w(k) = \sum_{m=0}^{g-1} \sum_{\substack{r \geq 0 \\ a_w(2gr+2m+1)=k}} L(2gr+2m+1).$$

Now we claim that

$$a_w(2gn+2m+1) - a_w(gn+m) = \begin{cases} 1 & \text{if } m = \lfloor v(w)/2 \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

This is easy to see, since the binary expansion of $2gn + 2m + 1$ is the same as that for $gn + m$, except that there is an extra 1-bit on the end. Hence if

$$a_w(2gn + 2m + 1) > a_w(gn + m)$$

then the last $|w|$ bits of $2g+2m+1$ must coincide with the string w, and so $m = \lfloor v(w)/2 \rfloor$, since w ends in 1. Thus we see that all but one of the terms in the sums for $d_w(k)$ and $e_w(k)$ are identical, and hence, on recalling $h = \lfloor v(w)/2 \rfloor$,

$$d_w(k) - e_w(k) = \sum_{\substack{r \geq 0 \\ a_w(gr+h)=k}} L(2gr+v(w)) \; - \sum_{\substack{r \geq 0 \\ a_w(gr+h)=k-1}} L(2gr+v(w)). \qquad (4)$$

Now if we could show that $d_w(k) = e_w(k)$ for $k > 0$, then it would follow from equation (4) that the value of the sum

$$\sum_{\substack{r\geq 0\\ a_w(gr+h)=k}} L(2gr+v(w))$$

is *independent* of k and hence equal to $d_w(0)-e_w(0)$. In fact, we now show that

$$d_w(k)=e_w(k)+E_k$$

where

$$E_k=\begin{cases}-1 & \text{if } k=0,\\ 0 & \text{if } k\geq 1.\end{cases}$$

For we have

$$\sum_{\substack{n\geq 1\\ a_w(n)=k}} L(n)=\sum_{\substack{n\geq 1\\ a_w(2n)=k}} L(2n)+\sum_{\substack{n\geq 0\\ a_w(2n+1)=k}} L(2n+1).$$

Hence

$$\begin{aligned}e_w(k)&=\sum_{\substack{n\geq 0\\ a_w(2n+1)=k}} L(2n+1)=\sum_{\substack{n\geq 1\\ a_w(n)=k}} (L(n)-L(2n))\\ &=(-E_k)+\sum_{\substack{n\geq 0\\ a_w(n)=k}} L(2n+1)=d_w(k)-E_k.\end{aligned}$$

This completes the proof of case I.

Case II: w ends in a 0, but $w\neq 0^j$.

First, we define functions d'_w and e'_w similar to those defined above:

$$d'_w(k)=\sum_{\substack{n\geq 1\\ a_w(n)=k}} L(2n)$$

and

$$e'_w(k)=\sum_{\substack{n\geq 1\\ a_w(2n)=k}} L(2n).$$

Following the method used for Case I, we easily find

$$d'_w(k)-e'_w(k)=\sum_{\substack{r\geq 0\\ a_w(gn+h)=k}} L(2gr+v(w))\ -\sum_{\substack{r\geq 0\\ a_w(gn+h)=k-1}} L(2gr+v(w)).$$

As before, we shall obtain the relation between $d'_w(k)$ and $e'_w(k)$. We have

$$\sum_{\substack{n\geq 1\\ a_w(n)=k}} L(n) = \Big(\sum_{\substack{n\geq 1\\ a_w(2n)=k}} L(2n)\Big) + \Big(\sum_{\substack{n\geq 1\\ a_w(2n+1)=k}} L(2n+1)\Big) + E_k.$$

Hence

$$\begin{aligned} e'_w(k) = \sum_{\substack{n\geq 1\\ a_w(2n)=k}} L(2n) &= (-E_k) + \sum_{\substack{n\geq 1\\ a_w(n)=k}} (L(n) - L(2n+1)) \\ &= (-E_k) + \sum_{\substack{n\geq 1\\ a_w(n)=k}} L(2n) = d'_w(k) - E_k. \end{aligned}$$

This completes the proof of Case II.

Case III: $w = 0^j$.

Left to the reader.

This completes the proof of Theorem 2. ■

IV. Some unusual power series.

We now modify Theorem 2 to obtain some unusual power series:

Theorem 3.

Let w be a string of 0's and 1's, and

$$g = 2^{|w|-1}, \quad h = \left\lfloor \frac{v(w)}{2} \right\rfloor,$$

and let X be a complex number with $|X| < 1$. Then

$$\sum_n X^{a_w(gn+h)} L(2gn + v(w)) = -\frac{1}{1-X},$$

where the sum is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise.

Proof.

For $|X| < 1$, the series $\sum_{n\geq 0} X^n$ is absolutely convergent, and the quantities

$$L(2gn + v(w))$$

are all negative; thus we have

$$
\begin{aligned}
-\frac{1}{1-X} &= \sum_{k\geq 0} X^k(-1) \\
&= \sum_{k\geq 0} X^k \sum_{\substack{n \\ a_w(gn+h)=k}} L(2gn+v(w)) \\
&= \sum_{n} X^{a_w(gn+h)} L(2gn+v(w)),
\end{aligned}
$$

where we have used Theorem 2. ■

This result, although appealing, is unsatisfactory in two ways. First, the exponent of X is $a_w(gn+h)$ instead of $a_w(n)$. Second, in order to obtain the infinite products mentioned in the introduction, we must show that Theorem 3 in fact holds for all $|X| \leq 1$, $X \neq 1$. The first of these problems is corrected in this section, while the question of convergence on the unit circle is addressed in the next section.

We now show how to modify Theorem 2 so that the summation is over all n with $a_w(n) = k$. To do this, we need the notion of a *suffix* of a string. Let x and y be two strings. Then we say x is a suffix of y if there exists a third string z such that $y = zx$.

Now we prove the following

Lemma 4.

Let t be an integer with binary expansion $t = b_1 b_2 \cdots b_r b_{r+1} \cdots b_s$.

(A) If $b_1 b_2 \cdots b_r$ is not a suffix of w, then

$$
\sum_{\substack{n \\ a_w(2^r n + v(b_1\cdots b_r))=k}} L(2^s n + t) = \sum_{\substack{n \\ a_w(2^{r-1} n + v(b_1\cdots b_{r-1}))=k}} L(2^s n + t).
$$

(B) If, however, $b_1 b_2 \cdots b_r$ is a suffix of w, then

$$
\sum_{\substack{n \\ a_w(2^r n + v(b_1\cdots b_r))=k}} L(2^s n + t) =
$$

$$
\sum_{\substack{n \\ a_w(2^{r-1} n + v(b_2\cdots b_r))=k}} L(2^{s-1} n + t_1) - \sum_{\substack{n \\ a_w(2^{r-1} n + v(\overline{b_1} b_2\cdots b_{r-1}))=k}} L(2^s n + t_2)
$$

where $t_1 = v(b_2 b_3 \cdots b_r b_{r+1} \cdots b_s)$, $t_2 = v(\overline{b_1} b_2 \cdots b_r b_{r+1} \cdots b_s)$, and by $\overline{b}$ we mean the complement of the bit b; i. e. $\overline{0} = 1; \overline{1} = 0$.

Proof.

Part (A) of the lemma can be left to the reader. To prove part (B), we start with the summation

$$\sum_{\substack{n \\ a_w(2^{r-1}n+v(b_2\cdots b_r))=k}} L(2^{s-1}n+t_1)$$

and break the sum into two parts, corresponding to $n = 2m + b_1$ and $n = 2m + \overline{b_1}$, where b_1 is a single bit. We get

$$\sum_{\substack{n \\ a_w(2^{r-1}n+v(b_2\cdots b_r))=k}} L(2^{s-1}n+t_1) =$$

$$\sum_{\substack{m \\ a_w(2^r m+v(b_1\cdots b_r))=k}} L(2^s m+t) + \sum_{\substack{m \\ a_w(2^r m+v(\overline{b_1}b_2\cdots b_r))=k}} L(2^s m+t_2).$$

Now $b_1 b_2 \cdots b_r$ is a suffix of w, so $\overline{b_1} b_2 \cdots b_r$ cannot be a suffix of w. Thus we may change the index of summation in the rightmost sum to $a_w(2^{r-1}m + v(\overline{b_1} b_2 \cdots b_{r-1}))$, and the result follows. ■

Lemma 5.

There is a rational function $b_w(n)$ such that for all $k \geq 0$ we have

$$\sum_{\substack{n \\ a_w(n)=k}} \log_2(b_w(n)) = -1. \tag{5}$$

(The summation is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise.) This function $b_w(n)$ is effectively computable, and the degree d_w of the numerator and denominator of $b_w(n)$ satisfies

$$d_w \leq 2^{|w|-1}.$$

Proof.

To prove the first statement, we start with Theorem 2 and successively apply Lemma 4. At each step, we convert a sum over $a_w(2^r n + x_1)$ to either one or two sums over $a_w(2^{r-1}n+x_2)$. Thus, after at most $|w|-1$ stages (and a total of at most $2^{|w|-1}$ invocations of Lemma 4), we will reduce the original sum (3) to a sum of sums of the form (5), which can be combined in a single sum using the properties of the logarithm. The degree d_w of the numerator and denominator correspond to the number of invocations of Lemma 4 (B), and hence $d_w \leq 2^{|w|-1}$. ■

Let us give an example. For $w = 1010$, Theorem 2 shows that

$$\sum_{\substack{n\geq 0\\ a_{1010}(8n+5)=k}} L(16n+10) = -1.$$

By successive applications of Lemma 4, we find

$$\sum_{\substack{n\geq 0\\ a_{1010}(8n+5)=k}} L(16n+10) = \sum_{\substack{n\geq 0\\ a_{1010}(n)=k}} \Big(L(4n+2) - L(8n+6) - L(8n+2) + L(16n+10)\Big).$$

Putting this all together, we conclude that, for all $k \geq 0$,

$$\sum_{\substack{n\geq 0\\ a_{1010}(n)=k}} \log_2\left(\frac{(4n+2)(8n+7)(8n+3)(16n+10)}{(4n+3)(8n+6)(8n+2)(16n+11)}\right) = -1.$$

Comment.

It can be shown that $\max_{|w|\leq x} d_w = O(x^{10.1})$; from this it easily follows that the algorithm to calculate $b_w(n)$ is actually a polynomial-time algorithm. For the details, see [4].

V. Behaviour On the Unit Circle.

In this section, we examine the convergence of

$$\sum_n X^{a_w(n)} L(2gn + v(w)) \tag{6}$$

on the unit circle. We will show that (6) converges *uniformly* on each radius of the unit disc, with the exception of the radius lying on the positive reals.

It suffices to show uniform convergence for the series

$$\sum_{n\geq 1} \frac{X^{a_w(n)}}{n}.$$

For this, we follow the technique used previously in [3]. Write

$$T_w(N) = T_w(N, X) = \sum_{0\leq n<N} X^{a_w(n)}$$

so that

$$\sum_{1\leq n\leq N} \frac{X^{a_w(n)}}{n} = \frac{T_w(N+1)}{N} - T_w(1) + \sum_{1\leq n\leq N-1} \frac{T_w(n+1)}{n(n+1)}.$$

Thus it suffices to show that, for each ray from the origin to a point ($\neq 1$) on the unit circle, there exists $\alpha < 1$ such that

$$|T_w(N, X)| = O(N^{\alpha}) \tag{7}$$

uniformly in X on the ray.

We do this for $w \neq 0^j$, leaving the case $w = 0^j$ to the reader. For $w = 1$, this easily follows from the fact that

$$T_1(2^m, X) = (X+1)^m.$$

Thus let us assume that $|w| \geq 2$. We write $A = 2^{|w|-1}$ and define A different sums, $P_0(m)$ through $P_{A-1}(m)$, by

$$P_i(m) = \sum_{0 \leq n < A^m} X^{a_w(An+i)}, \quad 0 \leq i < A.$$

Clearly at least one of the sums $P_i(m)$ coincides with $T_w(A^m)$; if $w = w_1 w_2 \cdots w_k$, we may take

$$i = v(\underbrace{\overline{w_k w_k} \cdots \overline{w_k}}_{k}).$$

Thus to bound $T_w(A^m)$, it suffices to give an upper bound for

$$\|P(m)\|_{\infty},$$

where the column vector $P(m)$ is defined by

$$P(m) = \begin{pmatrix} P_0(m) \\ \vdots \\ P_{A-1}(m) \end{pmatrix},$$

and the norm $\|v\|_{\infty}$ is defined by

$$\|v\|_{\infty} = \sum_{1 \leq i \leq k} |v_i|.$$

First, we observe that the vector $P(m)$ can be written as a linear transformation of the vector $P(m-1)$. Recall that $a'_w(z)$ counts the number of occurrences of the string w in the string z. Then we have the following

Lemma 6.

Let $M_w(X)$ be the $A \times A$ matrix with

$$M_w(X) = [M_{ij}] = \left[X^{a'_w(y_j y_i)}\right],$$

where y_j and y_i are strings such that $v(y_j) = j$, $v(y_i) = i$, $|y_j| = |y_i| = |w| - 1$. Then

$$P(m) = M_w(X)P(m-1).$$

Proof.

We have

$$\begin{aligned}
P_i(m) &= \sum_{0\le n<A^m} X^{a_w(An+i)} \\
&= \sum_{0\le j<A} \sum_{0\le k<A^{m-1}} X^{a_w(A(Ak+j)+i)} \\
&= \sum_{0\le j<A} \sum_{0\le k<A^{m-1}} X^{a_w(Ak+j)+a'_w(y_j y_i)} \\
&= \sum_{0\le j<A} X^{a'_w(y_j y_i)} P_j(m-1). \quad \blacksquare
\end{aligned}$$

Thus to bound $\|P(m)\|_\infty$, it suffices to determine a good bound for the matrix norm $\|M_w(X)\|_\infty$, where

$$\|M\|_\infty = \max_{1\le i\le n} \sum_{1\le j\le n} |M_{ij}|.$$

Lemma 7.

(A) The matrix $M_w(X)$ contains at least one row of all 1's.

(B) At least one row of $M_w(X)$ is not identically 1. In every such row, there is at least one element 1 and at least one element X.

Proof.

(A) Write $w = w_1 w_2 \cdots w_k$. Let

$$i' = v(\underbrace{\overline{w_k}\,\overline{w_k} \cdots \overline{w_k}}_{k-1}).$$

Then it is easily verified that w is not a substring of any string of the form $y_j y_{i'}$, and the result follows from the description of Lemma 6.

(B) To see that at least one row of $M_w(X)$ is not identically 1, let

$$i'' = v(w_2 w_3 \cdots w_k).$$

Then in row i'', any column j whose least significant bit equals w_1 will contain a non-1 entry.

An argument similar to (A), but for columns, shows that every row contains at least one element 1.

Finally, let i be the index of a row of $M = M_w(X)$ that is not identically 1. Then w must be a substring of some string of the form $y_j y_i$. Choose j so that this substring appears as far to the right as possible; then $y_j y_i = xwz$ for some strings x and z. Then either $|x| = 0$, in which case $M_{ij} = X$, or $|x| = p > 0$. In this latter case, let

$$x' = \underbrace{\overline{w_1}\,\overline{w_1} \cdots \overline{w_1}}_{p}.$$

Then w matches the string $x'wz$ in exactly one place, and this string corresponds to $y_{j'} y_i$ for some value of j'; i. e. an element X in the i'th row. ■

Now let us recall the basic facts about matrix norms [5]. Let M be a matrix. Let $\sigma(M)$ be the set of all eigenvalues of M. The *spectral radius* of M, $r_\sigma(M)$, is defined as

$$r_\sigma(M) = \max_{\lambda \in \sigma(M)} |\lambda|.$$

Define the matrix norm

$$\|M\|_2 = \sqrt{r_\sigma(MM^*)},$$

where M^* denotes the conjugate transpose.

Lemma 8.

Let $|X| \le 1, X \ne 1$. Then $\|M_w(X)\|_2 < A$.

Proof.

Let $|X| \le 1, X \ne 1$. Write $M = M_w(X)$ and $M' = MM^*$. Note that each element M_{ij} of M satisfies $|M_{ij}| \le 1$. Fix a row i of the matrix M. If it is identically 1, then by Lemma 7, there exists a column j of M^* which contains an element 1 and an element $\overline{X}$, the conjugate of X. Hence

$$|M'_{ij}| \le A - 2 + |\overline{X} + 1| < A.$$

If row i is not identically 1, then by Lemma 7, it must contain an element 1 and an element X. Let j' be a column of M^* which is identically 1. Then again $|M'_{ij'}| < A$.

Thus we see that each row of M' contains an element M'_{ij} with $|M'_{ij}| < A$. Thus for each i, $0 \le i < A$ we have

$$\sum_{0 \le j < A} |M'_{ij}| < A^2.$$

Thus $\|M'\|_\infty < A^2$. By a well-known theorem [5, Theorem 7.8],

$$r_\sigma(M') \le \|M'\|_\infty,$$

and the result follows. ■

Lemma 9.

There exists $\alpha < 1$ such that

$$|T_w(N, X)| = O(N^\alpha)$$

uniformly in X on any ray from the origin to a point ($\neq 1$) on the unit circle.

Proof.

Write $c(X) = \|M_w(X)\|_2$. Lemma 8 shows that $c(X) < A$. Fix an angle θ, $0 < \theta < 2\pi$, and consider $c(X)$ as a function of $X = re^{i\theta}$, $0 \leq r \leq 1$. Since $c(X)$ is a continuous function of r, and the interval $0 \leq r \leq 1$ is compact, $c(X)$ must attain its maximum on this interval, which is necessarily a positive constant $c < A$.

Thus there are constants c_1, c_2 such that

$$\begin{aligned} |T_w(A^m, X)| &\leq \|P(m)\|_\infty \leq c_1\|P(m)\|_2 \\ &= c_1\|M_w(X)^m P(0)\|_2 = c_1\|M_w(X)\|_2{}^m\|P(0)\|_2 \\ &\leq c_2 c^m, \end{aligned}$$

uniformly in X on the ray.

Now writing $\alpha = \frac{\log c}{\log A} < 1$, we have proved

$$|T_w(N, X)| = O(N^\alpha)$$

uniformly in X on the ray, for $N = A^m$. We now sketch how to extend this bound to *all* integers N, as was done previously in [2].

To do this, we observe that $T_w(N, X)$ can be written as a sum over at most $A \log_A N$ terms of the form

$$Q_{d,i}(m) = \sum_{d \cdot A^m \leq n < (d+1) \cdot A^m} X^{a_w(An+i)}.$$

It is easily verified that the vector

$$Q_d(m) = \begin{pmatrix} Q_{d,0}(m) \\ \vdots \\ Q_{d,A-1}(m) \end{pmatrix}$$

can be written as a linear transformation of the vector $Q_d(m-1)$, using the same matrix that appeared in Lemma 6. Repeating the argument in Lemmas 6-8, we see that $|Q_{d,i}(m)|$ is bounded in the same way as $|P_i(m)|$. Thus

$$|T_w(N, X)| = O((\log N)N^\alpha) = O(N^{\alpha'})$$

where $\alpha' < 1$. ■

We have shown

Theorem 10.

Theorem 3 also holds for all X on the unit circle except $X = 1$.

There is a rational function $b_w(n)$ (which is effectively computable using Lemmas 4 and 5) such that, for all $X \neq 1$ with $|X| \leq 1$, we have

$$\sum_n \log_2(b_w(n)) X^{a_w(n)} = -\frac{1}{1-X}.$$

(Here the summation is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise.)

Comment.

After the original version of this paper was completed, we learned of the results of Boyd, Cook, and Morton, in which they study the partial sums

$$\sum_{0 \leq n \leq N} (-1)^{a_w(n)}$$

for arbitrary $w \in (0+1)^*$. This corresponds to the case $X = -1$ in equation (7). See [6].

VI. Some consequences.

Let us put $w = 0$, $X = -1$ in Theorem 10 and exponentiate. We get a nice companion formula to (1); namely,

$$\prod_{n \geq 1} \left(\frac{2n}{2n+1}\right)^{(-1)^{a_0(n)}} = \frac{\sqrt{2}}{2}.$$

Similarly, using the example from section IV, we find

$$\prod_{n \geq 0} \left(\frac{(4n+2)(8n+7)(8n+3)(16n+10)}{(4n+3)(8n+6)(8n+2)(16n+11)}\right)^{(-1)^{a_{1010}(n)}} = \frac{\sqrt{2}}{2}.$$

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