Characteristic Words as Fixed Points of Homomorphisms

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Abstract.

With each real number θ, $0 < \theta < 1$, we can associate the so-called *characteristic word* $w = w(\theta)$, defined by

$$w_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor,$$

for $n \geq 1$. We prove the following: if θ has a purely periodic continued fraction expansion, then $w(\theta)$ is a fixed point of a certain homomorphism $\varphi = \varphi_\theta$.

I. Introduction.

Let θ be a real number, $0 < \theta < 1$. Many authors have studied the so-called *characteristic word* $w = w(\theta)$, the infinite word of `0`'s and `1`'s defined by

$$w_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor \tag{1}$$

for $n \geq 1$. See, for example, Bernoulli [1772], Markoff [1882], Venkov [1970, pp. 65-68], Stolarsky [1976], Fraenkel, Mushkin, and Tassa [1978], and Porta and Stolarsky [1990]. An extensive bibliography of papers on the subject can be assembled by consulting the references of the last three papers.

For example, if $\theta = \frac{1}{2}(\sqrt{5} - 1)$, we find

$$w = w_1 w_2 w_3 \cdots = \texttt{1011010110} \cdots, \tag{2}$$

the so-called *Fibonacci word.*

It is well-known that the Fibonacci word is the unique fixed point of the homomorphism φ, where $\varphi(\texttt{0}) = \texttt{1}$, $\varphi(\texttt{1}) = \texttt{10}$. For this and other properties see, for example, Berstel [1986].

In this note we generalize this characterization (fixed point of a homomorphism) of the Fibonacci word to the case where θ has a purely periodic continued fraction expansion, i.e. when

$$\theta = [0, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, \ldots].$$

We refer to the number r as the *period length* of θ.

II. The Main Result.

First, we introduce some notation. Let θ be an irrational number, $0 < \theta < 1$. Write

$$\theta = [0, a_1, a_2, a_3, \ldots].$$

We define

$$\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n].$$

Note that $q_0 = 1$, $q_1 = a_1$, and for $n \geq 2$ we have

$$q_n = a_n q_{n-1} + q_{n-2}. \qquad (3)$$

Let $w = w(\theta)$ be the characteristic word of θ as defined in (1) above.

We now define a sequence of strings $(X_i)_{i \geq 0}$. We set $X_0 = \mathtt{0}$, a string of length 1, and

$$X_i = w_1 w_2 w_3 \cdots w_{q_i}$$

for $i \geq 1$. Thus for $i \geq 1$, X_i consists of the first q_i symbols in the infinite word w. It is easy to see that $X_1 = \mathtt{0}^{a_1 - 1}\mathtt{1}$.

The following result essentially appears in the paper of Fraenkel, Mushkin and Tassa [1978]. Since it is crucial to our proof, and since it does not seem to have been explicitly stated before, we give it the status of a lemma:

Lemma 1.

For $i \geq 2$ we have

$$X_i = X_{i-1}^{a_i} X_{i-2}.$$

Proof.

Let us borrow a notation from the programming language APL. If $x = x_1 x_2 \cdots x_s$ is a finite string, and n is a non-negative integer, we define

$$n \rho x = x^q x_1 x_2 \cdots x_r,$$

where $n = qs + r$, $0 \leq r < s$. (In other words, the elements of x are used cyclically to fill in a string of length n.)

Fraenkel, Mushkin, and Tassa [1978] proved that

$$X_i = q_i \rho X_{i-1}$$

for $i \geq 2$, if $a_1 > 1$, and for $i \geq 3$ if $a_1 = 1$.

From this, the lemma follows immediately, since by (3) we have $q_i = a_i q_{i-1} + q_{i-2}$ for $i \geq 2$, and X_{i-2} is a prefix of X_{i-1} (for $i \geq 2$ if $a_1 > 1$ and for $i \geq 3$ if $a_1 = 1$). ■

We can now state the main result:

Theorem 2.

Let θ have a purely periodic continued fraction expansion; i.e.

$$\theta = [0, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, \ldots].$$

Define the homomorphism φ by $\varphi(\mathtt{0}) = X_r$, $\varphi(\mathtt{1}) = X_r X_{r-1}$. Then

$$\varphi^n(X_i) = X_{rn+i}$$

for all integers $i, n \geq 0$.

Proof.

By induction on $rn + i$.

If $rn + i = 0$, then $n = 0$ and $i = 0$. Clearly $\varphi^0(X_0) = X_0$.

If $rn + i = 1$, then either $n = 0$, $i = 1$, or $r = 1$, $n = 1$, and $i = 0$. In the former case we have $\varphi^0(X_1) = X_1$. In the latter case we have $\varphi(X_0) = \varphi(\mathtt{0}) = X_1$ by definition of φ.

Now assume the result is true for all n', i' with $rn' + i' < s$, and $s \geq 2$. We prove it for $rn + i = s$.

Case I: $i \geq 2$. We find

$$\begin{aligned}
\varphi^n(X_i) &= \varphi^n(X_{i-1}^{a_i} X_{i-2}) \quad \text{(by Lemma 1)} \\
&= \varphi^n(X_{i-1}^{a_i})\varphi^n(X_{i-2}) \\
&= \varphi^n(X_{i-1})^{a_i}\varphi^n(X_{i-2}) \\
&= X_{rn+i-1}^{a_i} X_{rn+i-2} \quad \text{(by induction)} \\
&= X_{rn+i} \quad \text{(by Lemma 1).}
\end{aligned}$$

Case II: $i = 1$, $n \geq 1$. We find

$$\begin{aligned}
\varphi^n(X_1) &= \varphi^{n-1}(\varphi(X_1)) \\
&= \varphi^{n-1}(\varphi(\mathtt{0}^{a_1-1}\mathtt{1})) \\
&= \varphi^{n-1}(\varphi(\mathtt{0})^{a_1-1}\varphi(\mathtt{1})) \\
&= \varphi^{n-1}(X_r^{a_1-1}X_rX_{r-1}) \\
&= \varphi^{n-1}(X_r^{a_1}X_{r-1}) \\
&= \varphi^{n-1}(X_r)^{a_1}\varphi^{n-1}(X_{r-1}) \\
&= X_{rn}^{a_1}X_{rn-1} \quad \text{(by induction)} \\
&= X_{rn+1} \quad \text{(by Lemma 1).}
\end{aligned}$$

Case III: $i = 0$, $n \geq 1$, $r \geq 2$. We find

$$\begin{aligned}
\varphi^n(X_0) &= \varphi^{n-1}(\varphi(X_0)) \\
&= \varphi^{n-1}(X_r) \\
&= \varphi^{n-1}(X_{r-1}^{a_r}X_{r-2}) \quad \text{(by Lemma 1)} \\
&= \varphi^{n-1}(X_{r-1})^{a_r}\varphi^{n-1}(X_{r-2}) \\
&= X_{rn-1}^{a_r}X_{rn-2} \quad \text{(by induction)} \\
&= X_{rn} \quad \text{(by Lemma 1).}
\end{aligned}$$

Case IV: $i = 0$, $n \geq 2$, $r = 1$. We find

$$\begin{aligned}
\varphi^n(X_0) &= \varphi^{n-2}(\varphi^2(X_0)) \\
&= \varphi^{n-2}(\varphi(X_1)) \\
&= \varphi^{n-2}(\varphi(\mathtt{0}^{a_1-1}\mathtt{1})) \\
&= \varphi^{n-2}(X_1)^{a_1-1}\varphi^{n-2}(X_1X_0) \\
&= X_{n-1}^{a_1-1}X_{n-1}X_{n-2} \text{ (by induction)} \\
&= X_n \quad \text{(by Lemma 1).}
\end{aligned}$$

This completes the proof. ■

Since in particular $X_{rn} = \varphi^n(X_0)$, we find

Corollary 3.

The infinite word w is a fixed point of the homomorphism φ defined above.

III. Some examples.

Example 1.

Let $\theta = [0, a, a, a, \ldots] = \frac{1}{2}(\sqrt{a^2+4} - a)$. Thus $r = 1$; we find $p_1/q_1 = 1/a$. Then we find $X_0 = \mathtt{0}$ and $X_1 = \mathtt{0}^{a-1}\mathtt{1}$. Thus $w(\theta)$ is a fixed point of the homomorphism φ, where $\varphi(\mathtt{0}) = \mathtt{0}^{a-1}\mathtt{1}$, $\varphi(\mathtt{1}) = \mathtt{0}^{a-1}\mathtt{10}$. For $a = 1$ this gives the classical Fibonacci word, mentioned in Section I.

Note that φ satisfies the equation

$$\varphi^2(0) = \varphi(0)^a 0,$$

and so is an "algebraic" homomorphism; see Shallit [1988].

Example 2.

Let $\theta = [0, a, b, a, b, \ldots] = (\sqrt{ab(ab+4)} - ab)/2a$. Thus $r = 2$; we find $p_1/q_1 = 1/a$ and $p_2/q_2 = b/(ab+1)$. Thus $X_0 = \mathtt{0}$, $X_1 = \mathtt{0}^{a-1}\mathtt{1}$, and $X_2 = (\mathtt{0}^{a-1}\mathtt{1})^b\mathtt{0}$. From this, we see that $w(\theta)$ is a fixed point of the homomorphism φ, where $\varphi(\mathtt{0}) = (\mathtt{0}^{a-1}\mathtt{1})^b\mathtt{0}$, $\varphi(\mathtt{1}) = (\mathtt{0}^{a-1}\mathtt{1})^b\mathtt{0}^a\mathtt{1}$.

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In what must be one of the more remarkable instances of simultaneous discovery of the same theorem, after this manuscript was completed, I learned from J.-P. Allouche of the work of T. C. Brown [1990] and J.-P. Borel and F. Laubie [1990]. These papers contain essentially the same result as I reported above in Theorem 2, and more. (However, I believe my proof of Theorem 2 to be simpler than Brown's.)

Furthemore, Allouche later discovered the paper of Ito and Yasutomi [1990], in which the same result appears. Then, in April 1991, at the "Thémate" Conference, I was given a preprint of Nishioka, Shiokawa, and Tamura [1991], in which the result appears once again!

In May 1991, in conversations with A. D. Pollington, I learned that some of these results can be found, in a somewhat concealed fashion, in a little-known paper of Cohn [1974]. Pollington himself has a paper [1991] on this topic!

I also discovered that Lemma 1 essentially already appeared in an little-known paper of H. J. S. Smith [1876].

Finally, Theorem 2 can be used to greatly simplify the proof of one direction of a beautiful theorem of F. Mignosi [1989].

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