Characteristic Words as Fixed Points of Homomorphisms

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Abstract.

With each real number $\theta$, $0 < \theta < 1$, we can associate the so-called characteristic word $w = w(\theta)$, defined by

$$w_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor,$$

for $n \geq 1$. We prove the following: if $\theta$ has a purely periodic continued fraction expansion, then $w(\theta)$ is a fixed point of a certain homomorphism $\varphi = \varphi_\theta$.

I. Introduction.

Let $\theta$ be a real number, $0 < \theta < 1$. Many authors have studied the so-called characteristic word $w = w(\theta)$, the infinite word of 0’s and 1’s defined by

$$w_n = \lfloor (n+1)\theta \rfloor - \lfloor n\theta \rfloor$$

for $n \geq 1$. See, for example, Bernoulli [1772], Markoff [1882], Venkov [1970, pp. 65-68], Stolarsky [1976], Fraenkel, Mushkin, and Tassa [1978], and Porta and Stolarsky [1990]. An extensive bibliography of papers on the subject can be assembled by consulting the references of the last three papers.

For example, if $\theta = \frac{1}{2}(\sqrt{5} - 1)$, we find

$$w = w_1w_2w_3 \cdots = 1011010110 \cdots,$$

the so-called Fibonacci word.

It is well-known that the Fibonacci word is the unique fixed point of the homomorphism $\varphi$, where $\varphi(0) = 1$, $\varphi(1) = 10$. For this and other properties see, for example, Berstel [1986].
In this note we generalize this characterization (fixed point of a homomorphism) of the Fibonacci word to the case where $\theta$ has a purely periodic continued fraction expansion, i.e. when

$$\theta = [0, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, \ldots].$$

We refer to the number $r$ as the *period length* of $\theta$.

II. The Main Result.

First, we introduce some notation. Let $\theta$ be an irrational number, $0 < \theta < 1$. Write

$$\theta = [0, a_1, a_2, a_3, \ldots].$$

We define

$$\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n].$$

Note that $q_0 = 1$, $q_1 = a_1$, and for $n \geq 2$ we have

$$q_n = a_n q_{n-1} + q_{n-2}. \quad (3)$$

Let $w = w(\theta)$ be the characteristic word of $\theta$ as defined in (1) above.

We now define a sequence of strings $(X_i)_{i \geq 0}$. We set $X_0 = 0$, a string of length 1, and

$$X_i = w_1 w_2 w_3 \cdots w_{q_i}$$

for $i \geq 1$. Thus for $i \geq 1$, $X_i$ consists of the first $q_i$ symbols in the infinite word $w$. It is easy to see that $X_1 = 0^{a_1-1} 1$.

The following result essentially appears in the paper of Fraenkel, Mushkin and Tassa [1978]. Since it is crucial to our proof, and since it does not seem to have been explicitly stated before, we give it the status of a lemma:

**Lemma 1.**

For $i \geq 2$ we have

$$X_i = X_i^{a_i} X_{i-2}.$$

**Proof.**

Let us borrow a notation from the programming language APL. If $x = x_1 x_2 \cdots x_n$ is a finite string, and $n$ is a non-negative integer, we define

$$n \rho x = x^q x_1 x_2 \cdots x_r,$$
where \( n = qs + r, 0 \leq r < s \). (In other words, the elements of \( x \) are used cyclically to fill in a string of length \( n \).)

Fraenkel, Mushkin, and Tassa [1978] proved that

\[
X_i = q_i \rho X_{i-1}
\]

for \( i \geq 2 \), if \( a_1 > 1 \), and for \( i \geq 3 \) if \( a_1 = 1 \).

From this, the lemma follows immediately, since by (3) we have \( q_i = a_i q_{i-1} + q_{i-2} \) for \( i \geq 2 \), and \( X_{i-2} \) is a prefix of \( X_{i-1} \) (for \( i \geq 2 \) if \( a_1 > 1 \) and for \( i \geq 3 \) if \( a_1 = 1 \)).

We can now state the main result:

**Theorem 2.**

Let \( \theta \) have a purely periodic continued fraction expansion; i.e.

\[
\theta = [0, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, \ldots].
\]

Define the homomorphism \( \varphi \) by \( \varphi(0) = X_r \), \( \varphi(1) = X_r X_{r-1} \). Then

\[
\varphi^n(X_i) = X_{rn+i}
\]

for all integers \( i, n \geq 0 \).

**Proof.**

By induction on \( rn + i \).

If \( rn + i = 0 \), then \( n = 0 \) and \( i = 0 \). Clearly \( \varphi^0(X_0) = X_0 \).

If \( rn + i = 1 \), then either \( n = 0 \), \( i = 1 \), or \( r = 1 \), \( n = 1 \), and \( i = 0 \). In the former case we have \( \varphi^0(X_1) = X_1 \). In the latter case we have \( \varphi(X_0) = \varphi(0) = X_1 \) by definition of \( \varphi \).

Now assume the result is true for all \( n', i' \) with \( rn' + i' < s \), and \( s \geq 2 \). We prove it for \( rn + i = s \).

Case I: \( i \geq 2 \). We find

\[
\varphi^n(X_i) = \varphi^n(X_{i-1}^{a_i} X_{i-2}) \quad \text{(by Lemma 1)}
\]

\[
= \varphi^n(X_{i-1}^{a_i}) \varphi^n(X_{i-2})
\]

\[
= \varphi^n(X_{i-1})^{a_i} \varphi^n(X_{i-2})
\]

\[
= X_{rn+i-1}^{a_i} X_{rn+i-2} \quad \text{(by induction)}
\]

\[
= X_{rn+i} \quad \text{(by Lemma 1)}.
\]
Case II: $i = 1, n \geq 1$. We find
\[
\varphi^n(X_1) = \varphi^{n-1}(\varphi(X_1)) \\
= \varphi^{n-1}(\varphi(0^{a_1-1}1)) \\
= \varphi^{n-1}(\varphi(0)^{a_1-1}\varphi(1)) \\
= \varphi^{n-1}(X_r^{a_1-1}X_rX_{r-1}) \\
= \varphi^{n-1}(X_r^{a_1}X_{r-1}) \\
= \varphi^{n-1}(X_r)^{a_1}\varphi^{n-1}(X_{r-1}) \\
= X_r^{a_1}X_{r_n-1} \quad \text{(by induction)} \\
= X_{r_{n+1}} \quad \text{(by Lemma 1)}.
\]

Case III: $i = 0, n \geq 1, r \geq 2$. We find
\[
\varphi^n(X_0) = \varphi^{n-1}(\varphi(X_0)) \\
= \varphi^{n-1}(X_r) \\
= \varphi^{n-1}(X_r^{a_{r-1}}X_{r-2}) \quad \text{(by Lemma 1)} \\
= \varphi^{n-1}(X_{r-1})^{a_r}\varphi^{n-1}(X_{r-2}) \\
= X_r^{a_{r-1}}X_{r_n-2} \quad \text{(by induction)} \\
= X_{r_n} \quad \text{(by Lemma 1)}.
\]

Case IV: $i = 0, n \geq 2, r = 1$. We find
\[
\varphi^n(X_0) = \varphi^{n-2}(\varphi^2(X_0)) \\
= \varphi^{n-2}(\varphi(X_1)) \\
= \varphi^{n-2}(\varphi(0^{a_1-1}1)) \\
= \varphi^{n-2}(X_1)^{a_1-1}\varphi^{n-2}(X_1X_0) \\
= X_{n-1}^{a_1-1}X_{n-1}X_{n-2} \quad \text{(by induction)} \\
= X_n \quad \text{(by Lemma 1)}.
\]

This completes the proof. □

Since in particular $X_{rn} = \varphi^n(X_0)$, we find

**Corollary 3.**
The infinite word \( w \) is a fixed point of the homomorphism \( \varphi \) defined above.

III. Some examples.

Example 1.

Let \( \theta = [0, a, a, a, \ldots] = \frac{1}{2}(\sqrt{a^2 + 4} - a) \). Thus \( r = 1 \); we find \( p_1/q_1 = 1/a \). Then we find \( X_0 = 0 \) and \( X_1 = 0^{a-1}1 \). Thus \( w(\theta) \) is a fixed point of the homomorphism \( \varphi \), where \( \varphi(0) = 0^{a-1}1 \), \( \varphi(1) = 0^{a-1}10 \). For \( a = 1 \) this gives the classical Fibonacci word, mentioned in Section I.

Note that \( \varphi \) satisfies the equation

\[
\varphi^2(0) = \varphi(0)^a0,
\]

and so is an “algebraic” homomorphism; see Shallit [1988].

Example 2.

Let \( \theta = [0, a, b, a, b, \ldots] = (\sqrt{ab(ab + 4) - ab})/2a \). Thus \( r = 2 \); we find \( p_1/q_1 = 1/a \) and \( p_2/q_2 = b/(ab + 1) \). Thus \( X_0 = 0 \), \( X_1 = 0^{a-1}1 \), and \( X_2 = (0^{a-1}1)^b0 \). From this, we see that \( w(\theta) \) is a fixed point of the homomorphism \( \varphi \), where \( \varphi(0) = (0^{a-1}1)^b0 \), \( \varphi(1) = (0^{a-1}1)^b0^{a-1}1 \).

References

Bernoulli [1772]


Berstel [1986]


Fraenkel, Mushkin, and Tassa [1978]


Markoff [1882]


Porta and Stolarsky [1990]

Shallit [1988]


Stolarsky [1976]


Venkov [1970]


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In what must be one of the more remarkable instances of simultaneous discovery of the same theorem, after this manuscript was completed, I learned from J.-P. Allouche of the work of T. C. Brown [1990] and J.-P. Borel and F. Laubie [1990]. These papers contain essentially the same result as I reported above in Theorem 2, and more. (However, I believe my proof of Theorem 2 to be simpler than Brown’s.)

Furthermore, Allouche later discovered the paper of Ito and Yasutomi [1990], in which the same result appears. Then, in April 1991, at the “Thémate” Conference, I was given a preprint of Nishioka, Shiokawa, and Tamura [1991], in which the result appears once again!

In May 1991, in conversations with A. D. Pollington, I learned that some of these results can be found, in a somewhat concealed fashion, in a little-known paper of Cohn [1974]. Pollington himself has a paper [1991] on this topic!

I also discovered that Lemma 1 essentially already appeared in an little-known paper of H. J. S. Smith [1876].

Finally, Theorem 2 can be used to greatly simplify the proof of one direction of a beautiful theorem of F. Mignosi [1989].

\section*{Additional References}

Borel and Laubie [1991]


Brown [1991]

Cohn [1974]


Ito and Yasutomi [1990]


Mignosi [1989]


Pollington [1991]


Smith [1876]