

The Last Exception to a Suffix-Exponent Bound for the Fibonacci Word

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October 3 2026

Let $F_0 = 0$, $F_1 = 1$, and $F_{r+2} = F_{r+1} + F_r$. Let $\mathbf{f} = 01001010 \dots$ be the infinite Fibonacci word, indexed from 0. For a nonempty finite word w , its exponent is $|w|/\text{per}(w)$, where $\text{per}(w)$ is its least positive period. For $n \geq 1$, define

$$e(n) = \max \left\{ \frac{|w|}{\text{per}(w)} : w \text{ is a nonempty suffix of } \mathbf{f}[0..n-1] \right\}.$$

Theorem 1. *For every integer $k \geq 6$,*

$$\max \left\{ n \geq 1 : e(n) < \frac{F_{k+1} - 1}{F_{k-1}} \right\} = F_{2k-1} - F_k - 1.$$

Proof. Put

$$q_r = \frac{F_r - 1}{F_{r-2}} \quad (r \geq 3), \quad T = q_{k+1}, \quad N = F_{2k-1} - F_k - 1,$$

and write $\varphi = (1 + \sqrt{5})/2$. The identity

$$q_{r+1} - q_r = \frac{F_{r-3} + (-1)^{r+1}}{F_{r-1}F_{r-2}}$$

shows that q_r is nondecreasing, strictly increasing for $r \geq 5$, and converges to φ^2 . In particular,

$$2 < q_k < T < \varphi^2. \tag{1}$$

We first prove that $e(N) < T$. Define finite Fibonacci words by $X_1 = 1$, $X_2 = 0$, and $X_r = X_{r-1}X_{r-2}$ for $r \geq 3$. Thus $X_r = \mathbf{f}[0..F_r - 1]$ for $r \geq 2$. We use the elementary property that deleting the last two letters of X_r gives a palindrome. For completeness, the words X_rX_{r-1} and $X_{r-1}X_r$ agree except in their final two letters; this follows by induction from the defining recurrence. If C_r is X_r without its final two letters, and X_r ends in ab , then X_{r-1} ends in ba and

$$C_{r+1} = C_r ab C_{r-1} = C_{r-1} ba C_r.$$

Starting with C_3 empty and $C_4 = 0$, induction now shows that every C_r is a palindrome.

Reflecting a suffix of $\mathbf{f}[0..N-1]$ inside the palindrome of length $F_{2k-1} - 2$ gives an equally long factor, with the same least period, starting at

$$(F_{2k-1} - 2) - N = F_k - 1.$$

Thus it suffices to bound the exponents of prefixes of $\mathbf{f}[F_k - 1..\infty)$ of length at most N .

Every factor of the Fibonacci word has a Fibonacci number as its least period [1, Corollary 4]. We claim that a prefix of $\mathbf{f}[F_k - 1.. \infty)$ of length ℓ and period F_m satisfies

$$\ell \leq \begin{cases} F_{m+2} - 1, & 2 \leq m \leq k - 2, \\ F_m, & m = k - 1, \\ F_{m+2} - F_k - 1, & m \geq k. \end{cases} \quad (2)$$

Here is a direct verification. The last two letters of X_r give

$$\mathbf{f}[F_r - 1] = r \bmod 2, \quad \mathbf{f}[F_r - 2] = 1 - (r \bmod 2),$$

where the first identity holds for $r \geq 2$ and the second for $r \geq 3$. We also have

$$\mathbf{f}[F_u + F_v - 2] = \mathbf{f}[F_v - 2] \quad (3 \leq v \leq u). \quad (3)$$

For $v < u$, this follows from $X_{u+1} = X_u X_{u-1}$. For $v = u$, use $X_{u+2} = X_u X_{u-1} X_u$: for $u \geq 5$, the left side becomes $\mathbf{f}[F_{u-2} - 2]$, which has the required value; $u = 3, 4$ are checked directly.

If $m \leq k - 2$, equation (3) shows that the letters at

$$F_k + F_{m+1} - 2 \quad \text{and} \quad F_k + F_{m+2} - 2$$

differ. Their distance is F_m , proving the first bound in (2). If $m = k - 1$, the letters at $F_k - 1$ and $F_{k+1} - 1$ differ, proving the second bound. If $m \geq k$, the letters at $F_{m+1} - 2$ and $F_{m+2} - 2$ differ, proving the third bound.

For $m \leq k - 2$, equation (2) gives $\ell/F_m \leq q_{m+2} \leq q_k$. For $k \leq m \leq 2k - 3$, the quantities

$$\frac{F_{m+2} - F_k - 1}{F_m}$$

increase with m , since their consecutive differences have positive numerator

$$(F_k + 1)F_{m-1} + (-1)^{m+1}.$$

Their largest value in this range is N/F_{2k-3} . The case $m = k - 1$ gives exponent at most 1. Finally, for $m \geq 2k - 2$,

$$\frac{\ell}{F_m} \leq \frac{N}{F_{2k-2}} < 2.$$

Consequently,

$$e(N) \leq \max \left\{ q_k, \frac{N}{F_{2k-3}} \right\}. \quad (4)$$

Set

$$a = F_{k-1}, \quad b = F_{k-2}, \quad s = (-1)^k.$$

The standard determinant identity

$$F_u F_{v+2} - F_{u+2} F_v = (-1)^v F_{u-v} \quad (5)$$

and $F_{2k-3} = a^2 + b^2$ give

$$aN - (F_{k+1} - 1)F_{2k-3} = -b(a - b) - a + sb < 0.$$

Hence $N/F_{2k-3} < T$. Together with (1) and (4), this proves

$$e(N) < T. \quad (6)$$

It remains to prove that every $n > N$ satisfies $e(n) \geq T$. We use the established classification and suffix bounds from [2, Section 2 and the proof of Theorem 5]. If $n \geq 2$ and $e(n) < \varphi^2$, then n has one of the following forms:

1. $n = F_i - F_j - 1$, where $i \geq 5$ and $3 \leq j \leq i - 2$, with

$$e(n) \geq \max \left\{ \frac{n}{F_{i-2}}, q_j \right\}. \quad (7)$$

2. $n = F_i - F_j$, where $i \geq 5$, $3 \leq j \leq i - 2$, and j is odd, with

$$e(n) \geq \frac{n}{F_{i-2}}, \quad e(n) \geq \frac{F_j}{F_{j-2}} \quad \text{if } j \geq 5. \quad (8)$$

For the needed comparisons, define

$$H_i = F_{i-2} + sF_{i-k-1} \quad (i \geq 2k - 1).$$

Equation (5) gives

$$aF_i - (F_{k+1} - 1)F_{i-2} = H_i.$$

Thus, when $n = F_i - d$,

$$\frac{n}{F_{i-2}} \geq T \iff H_i \geq ad. \quad (9)$$

Moreover, H_i increases, because

$$H_{i+1} - H_i = F_{i-3} + sF_{i-k-2} > 0.$$

Its two relevant initial bounds are

$$H_{2k-1} \geq a(a+1), \quad H_{2k} \geq a(F_k + 1). \quad (10)$$

Indeed,

$$H_{2k-1} - a(a+1) = b^2 + sb - a \geq 0,$$

and

$$H_{2k} - a(F_k + 1) = a(b + s - 1) \geq 0,$$

using $b \geq 3$ and $b < a < 2b$.

Fix $n > N$. If $e(n) \geq \varphi^2$, we are done. Otherwise apply the classification above. Since $N > F_{2k-2}$ for $k \geq 6$, necessarily $i \geq 2k - 1$.

In the first form, $n = F_i - F_j - 1$, if $j \geq k + 1$, then (7) gives $e(n) \geq q_j \geq T$. Otherwise $j \leq k$. If $i = 2k - 1$, the inequality $n > N$ forces $j \leq k - 1$, so $d = F_j + 1 \leq a + 1$. If $i \geq 2k$, then $d = F_j + 1 \leq F_k + 1$. In both cases, equation (10), monotonicity of H_i , and equation (9) give $e(n) \geq T$.

In the second form, $n = F_i - F_j$ with j odd, first suppose $j \geq k - 1$. The ratios F_j/F_{j-2} increase along odd indices, and

$$\begin{aligned} \frac{F_{k-1}}{F_{k-3}} - T &= \frac{F_{k-3} + (-1)^{k-1}}{aF_{k-3}} > 0, \\ \frac{F_k}{F_{k-2}} - T &= \frac{F_{k-2} + (-1)^k}{aF_{k-2}} > 0. \end{aligned}$$

The smallest odd index at least $k - 1$ is either $k - 1$ or k . Therefore (8) gives $e(n) \geq F_j/F_{j-2} > T$. If instead $j \leq k - 2$, then $d = F_j \leq b < a + 1$, so equations (9)–(10) again give $e(n) \geq T$.

Thus every $n > N$ has $e(n) \geq T$, while (6) gives $e(N) < T$. This proves the theorem. $\square$

References

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- [2] J. Shallit, *Prefixes of the Fibonacci word*, arXiv:2302.04640v2 (2023). <https://arxiv.org/abs/2302.04640v2>.