Open questions
related to generalized palindromic richness

Š. Starosta

Czech Technical University in Prague

April 23, 2013

Reversal mapping and its fixed points

reversal mapping R

$$R(w_0 w_1 \dots w_n) = w_n \dots w_1 w_0$$

palindrome $w = R(w)$

examples: $0, 00, 010$

Palindromic richness

for $w \in \mathcal{A}^*$ we have

$$\#\mathrm{Pal}(w) \leq |w| + 1$$

(we count the empty word as a palindrome)

palindromic defect $D(w) = |w| + 1 - \#\mathrm{Pal}(w)$

for an infinite word $\mathbf{u}$ we define

$$D(\mathbf{u}) = \sup\{D(w) \mid w \text{ is a factor of } \mathbf{u}\}$$

a word is **rich/full** if its defect is 0

Θ-palindromes / pseudopalindromes / generalized palindromes

let $\Theta : \mathcal{A}^* \mapsto \mathcal{A}^*$ be an involutive antimorphism, i.e., $\Theta^2 = \mathrm{Id}$ and $\Theta(wv) = \Theta(v)\Theta(w)$ for all $w, v \in \mathcal{A}^*$

Θ-**palindrome** $w = \Theta(w)$

Example on $\mathcal{A} = \{0, 1\}$: $E : 0 \mapsto 1, 1 \mapsto 0$

E-palindromes: ε, 01, 10, 0101, 0011, ...

Θ-palindromes / pseudopalindromes / generalized palindromes

let $\Theta : \mathcal{A}^* \mapsto \mathcal{A}^*$ be an involutive antimorphism, i.e., $\Theta^2 = \mathrm{Id}$ and $\Theta(wv) = \Theta(v)\Theta(w)$ for all $w, v \in \mathcal{A}^*$

Θ-**palindrome** $w = \Theta(w)$

Example on $\mathcal{A} = \{0, 1\}$: $E : 0 \mapsto 1, 1 \mapsto 0$

E-palindromes: ε, 01, 10, 0101, 0011, ...

More symmetries

Let $\mathbf{u}$ be an infinite word over $\mathcal{A}$.

Let G be a **finite** group consisting of morphisms and antimorphisms over $\mathcal{A}$ such that the set of all factors of $\mathbf{u}$ is invariant under all elements of G.

A finite word w is a G-palindrome if there exists an antimorphism $\nu \in G$ such that $\nu(w) = w$.

$[w] = \{\nu(w) \mid \nu \in G\}$

G-defect

Let w be a finite word. The G-**defect** of w is

$$D_G(w) = |w| + 1 - \gamma_G(w) - \#\mathrm{Pal}_G(w),$$

where

$$\mathrm{Pal}_G(w) = \{[v] \mid v \text{ is a factor of } w \text{ and a } G\text{-palindrome}\}$$

and

$$\gamma_G(w) = \#\{[a] \mid a \in \mathcal{A}, a \text{ occurs in } w,\ a \neq \Theta(a) \text{ for all ant. } \Theta \in G\}$$

Let $\mathbf{u}$ be an infinite word. The G-**defect** of $\mathbf{u}$ is

$$D_G(\mathbf{u}) = \sup\{D_G(w) \mid w \text{ is a factor of } \mathbf{u}\}.$$

G-defect

Let w be a finite word. The G-**defect** of w is

$$D_G(w) = |w| + 1 - \gamma_G(w) - \#\mathrm{Pal}_G(w),$$

where

$$\mathrm{Pal}_G(w) = \{[v] \mid v \text{ is a factor of } w \text{ and a } G\text{-palindrome}\}$$

and

$$\gamma_G(w) = \#\{[a] \mid a \in \mathcal{A}, a \text{ occurs in } w,\ a \neq \Theta(a) \text{ for all ant. } \Theta \in G\}$$

Let $\mathbf{u}$ be an infinite word. The G-**defect** of $\mathbf{u}$ is

$$D_G(\mathbf{u}) = \sup\{D_G(w) \mid w \text{ is a factor of } \mathbf{u}\}.$$

Thue-Morse word

The Thue-Morse word is not rich, i.e., $\{R, \mathrm{Id}\}$-rich.

Its set of factors is closed under all elements of $H = \{R, E, \mathrm{Id}, E \cdot R\}$.

Its H-defect is 0 - it is H-rich.

$\gamma_H(w) = 0$ for all its factors w.

p	$\mathrm{Pal}_G(p)$
ε	$\{[\varepsilon]\}$
0	$\{[\varepsilon], [0]\}$
01	$\{[\varepsilon], [0], [01]\}$
011	$\{[\varepsilon], [0], [01], [11]\}$
0110	$\{[\varepsilon], [0], [01], [11], [0110]\}$

Thue-Morse word

The Thue-Morse word is not rich, i.e., $\{R, \mathrm{Id}\}$-rich.

Its set of factors is closed under all elements of $H = \{R, E, \mathrm{Id}, E \cdot R\}$.

Its H-defect is 0 - it is H-rich.

$\gamma_H(w) = 0$ for all its factors w.

p	$\mathrm{Pal}_G(p)$
ε	$\{[\varepsilon]\}$
0	$\{[\varepsilon], [0]\}$
01	$\{[\varepsilon], [0], [01]\}$
011	$\{[\varepsilon], [0], [01], [11]\}$
0110	$\{[\varepsilon], [0], [01], [11], [0110]\}$

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?

Open questions

1. What are examples of (classes of) G-rich words? So far, the generalized Thue-Morse words were proven to be G-rich [S, 2011].
2. Given a group G generated by involutive antimorphisms, is there a G-rich word?
3. Is a word having finite G_1-defect related to a G_2-rich word for some group G_2 isomorphic to G_1? In the classical sense [Bucci, De Luca, 2009; Pelantová, S. 2011]
4. Conjecture: if an infinite word is a fixed point of primitive injective morphism, then its G-defect is 0 or $+\infty$. This conjecture is still open for the classical notion of palindromic defect. (See open question 9 by M. Bucci.)
5. How many G-rich words of length n exist? (See open question 6 by A. Glen.)
6. (Added yesterday) Is the language of all G-rich words regular? Context-free?