# Kolakoski/Oldenburger word

$$\kappa = 122112122122112112212112122\ldots$$

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Let $W_k$ be the smallest code such that every $w \in W_k$ "gives" the same factor in $W_{k-1}^{+}$ after an integration.

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Let $W_k$ be the smallest code such that every $w \in W_k$ "gives" the same factor in $W_{k-1}^+$ after an integration.

Let $W' = \{\Delta(u) : u \in W_k^+, \Delta(u) \in C^\infty, u[1] = 1 \text{ and } u[|u|] = 2\}$. $W_{k+1}$ is the set of minimal elements in $W'$ by the prefix order.

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| | | | | | | | | | | | | | | | | | | |
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| 1 2 2 | 1 1 2 | 1 2 2 | 1 2 2 | 1 1 2 | 1 1 2 2 | 1 2 | 1 1 2 | 1 2 2 | 1 1 2 | 1 1 2 | 1 2 2 | 1 2 2 | 1 1 2 | 1 2 2 | 1 2 | 1 1 2 | 1 1 2 2 | ... |
| 12 | 21 | 12 | 12 | 21 | 22 | 11 | 21 | 12 | 21 | 21 | 12 | 12 | 21 | 12 | 11 | 21 | 22 | ... |
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$\Rightarrow$ $W_k$ is a bifix code.

# $W_k$ codes

| | |
|---|---|
| $W_1$ | $\{11, 12, 21, 22\}$ |
| $W_2$ | $\{11, 121121, 12112212, 1212, 1221, 2112, 2121, 21221121, 212212, 22\}$ |
| $W_3$ | $\{112112, 1121221121, 11212212, 1122121211221, 1122121212211,$<br>$112212121221221, 11221221, 121121, 1211221211, 12112212112,$<br>$12112212212112112212, 12112212212112122112, 121122122121211221,$<br>$1212, 1221121212211, 12211211212212112212, 122112121221221,$<br>$12211212212, 12211212212112112212, 122112122121211221,$<br>$122112121221221121, 12212112, 12212211, 1221221211211221,$<br>$1221221211212211, 12212212112212, 211211, 21121221,$<br>$211221211212112212, 211221211212122112, 2112212112122121221121,$<br>$21122121221121, 2112212212, 2121, 21221121212122112,$<br>$212211211212221211221, 212211211212221221121, 21221211221,$<br>$212211212212121211221, 2122112122121212112212, 2122112121221221,$<br>$21221211, 2122122112, 22\}$ |

# Smooth words ( = $C^\infty$ )

$\Delta$ is the *Run length encoding*.
Ex: $\Delta(\epsilon) = \epsilon$, $\Delta(12221) = 131$, $\Delta(121122) = 1122$.

$\kappa$ is the infinite word such that $\Delta(\kappa) = \kappa$.

$\delta(w)$ is $\Delta(w)$ without the first and/or last letter, if it is 1.
Ex: $\delta(11) = 2$, $\delta(12) = \epsilon$, $\delta(121) = 1$, $\delta(121122) = 122$

A word $w$ is *smooth* (or $C^\infty$) if for every $k \geq 0$ , $\delta^k(w) \in \{1,2\}^*$.

It is conjectured that the set of factors of $\kappa$ is $C^\infty$.

# Bifix Codes

A *bifix code* is a prefix code and a suffix code.

It is *maximal* if there is no other code which properly contains it.

Bifix codes have many nice properties.

$W_k$ is a bifix code, maximal in smooth words.

# Degree of Bifix codes

(In the general case, $F = \Sigma^*$)

The *degree* $d(X)$ of a maximal bifix code $X$ is the number of ways to decompose a bi-infinite word with $X$.

- this degree is independent of the word.

Let: $\pi(w) = \frac{1}{(|\Sigma|)^{|w|}}$.

If $X$ is a maximal bifix code, then :

$$\sum_{w \in X} |w| \cdot \pi(w) = d(X).$$

# Bifix codes on smooth words

Some results on bifix codes are generalized on languages $L$ s.t.:

- $L$ is factorial.
- $L$ is recurrent (*i.e.*: $\forall u, v \in L, \exists w$ s.t. $uwv \in L$).
- there is a $\pi : L \to \mathbb{R}^+$ such that $\sum_{x \in \Sigma} \pi(wx) = \sum_{x \in \Sigma} \pi(xw) = \pi(w)$.

see [Berstel, De Felice, Perrin, Reutenauer & Rindone 2010].

On smooth words:

- $C^\infty$ is factorial.
- The Dekking measure has the good property for $\pi$.
- The transitivity is conjectured:

**Conjecture (Universal Glueing Conjecture, Fédou & Fici 2010)**

*For every $u, v \in C^\infty$, $\exists w$ such that $uwv \in C^\infty$.*

# Dekking Measure

**[Dekking 95]**

For $w \in C^\infty$, let

$$\pi(w) = \frac{1}{s(w) \cdot 3^{h(w)}}$$

where:

- $s(\epsilon) = 1$
- $s(1) = s(2) = 2$
- $s(w) = s(\delta(w))$

- $h(1) = h(2) = h(\epsilon) = 0$
- $h(w) = 1 + h(\delta(w))$

**Question**

*Is there an infinite bifix code on smooth words ?*

Let $d(X) = \sum_{w \in X} |w| \pi(w)$.

**Question**

*Is it possible to prove $d(X) \in \mathbb{N}$ without UGC ?*

**Question**

*Is there an infinite bifix code on $C^\infty$ such that $d(X) < \infty$ ?*

**Question**

*Is it possible to prove $d(W_k) \in \mathbb{N}$ without UGC ?*

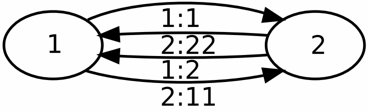
**Conjecture**

*For every $k$, $d(W_k) = 2^k$.*

(Tested up to $k = 8$.)

**Question**

*Is $W_k$ finite for every $k$ ?*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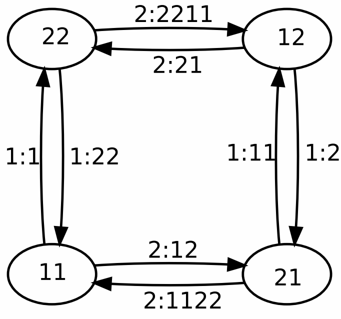



Figure: $\mathbb{T}^2$.

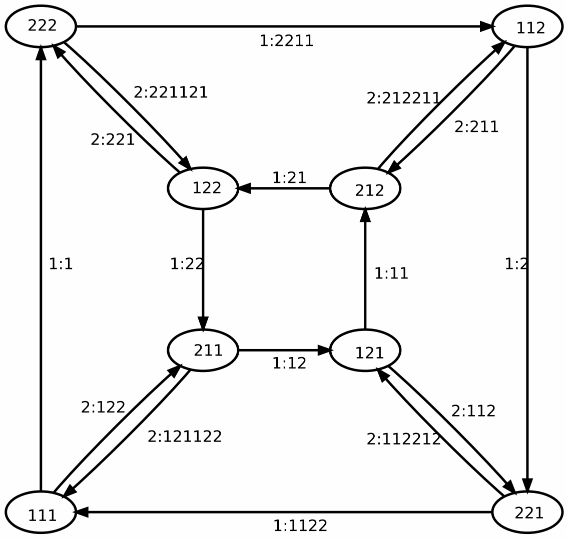
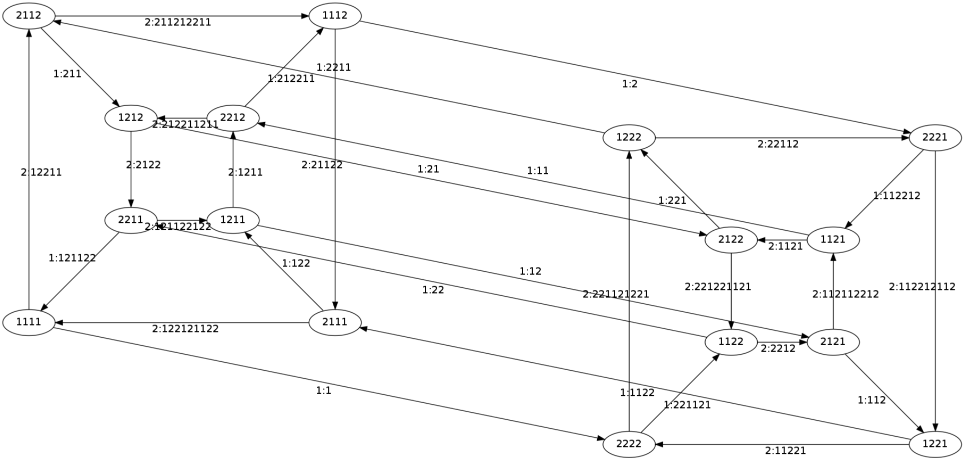



Figure: $\mathbb{T}^3$.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1122
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2121
1:1
1:1122
1:221121
1:112
2222
2:11221
1221

# Autres points fixes des $\mathbb{T}^k$

- $\kappa$ est également un point fixe de $\mathbb{T}^k$.
- Mais $\kappa^k$ avec $k \geq 2$ a d'autres points fixes !
- $\mathbb{T}^2$ a 4 points fixes: $\kappa$, $\kappa'$, $\kappa_2 = 1121122122121122122121\ldots$ et $\kappa_2' = 2122121122122112112122 11\ldots$ (qui sont les deux points limites de la séquence A111090).
- Presque toutes les questions ouvertes sur $\kappa$ restent pertinentes sur les autres point fixes de $\mathbb{T}^n$.
- Ces autres points fixes sont des cas spéciaux de *mots lisses infinis*.

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Exemples pour $\Sigma = \{a, b\}$ et $F = \Sigma^*$:

- $\{a, b\}$, $\{aa, ab, ba, aa\}$, $\{aaa, aab, aba, abb, baa, bab, bba, bbb\}$.
- $\{aaa, aaba, aabb, ab, baa, baba, babb, bba, bbb\}$
- $\{a\} \cup \{ba^k b\}_{k \geq 0}$.

Les codes bifixes ont étés étudiés généreusement dans le cas général (*càd* $F = \Sigma^*$) [Schützenberger *et al*, $\geq$ 1956].

Des recherches plus récentes cherchent à généraliser les résultats connus dans le cas général aux langages factoriels.

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