Iterated Hairpin Completion

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Challenges in Combinatorics on Words
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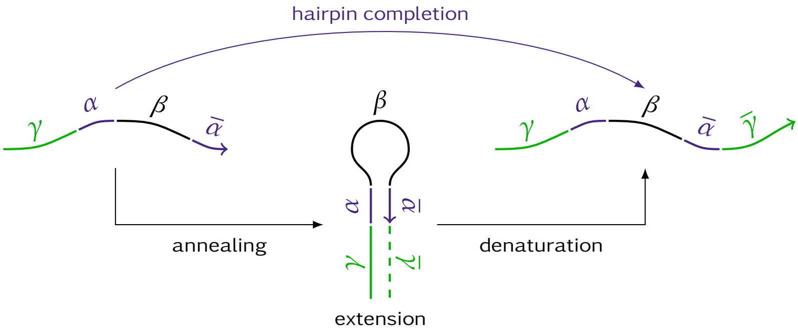
Hairpin Completion



Alphabet Σ

Complement $\overline{} : \Sigma \to \Sigma$ with $\overline{a_1 \cdots a_m} = \overline{a_m} \cdots \overline{a_1}$

Hairpin completion

$$\gamma\alpha\beta\bar{\alpha} \to \gamma\alpha\beta\bar{\alpha}\bar{\gamma} \qquad\qquad \alpha\beta\bar{\alpha}\bar{\gamma} \to \gamma\alpha\beta\bar{\alpha}\bar{\gamma}$$

Iterated hairpin completion

$$\mathcal{H}^*(w) = \{v \mid w \to^* v\}$$

Example

$$\alpha b \alpha c \alpha \overline{\alpha} \overline{d} \overline{\alpha} \overline{b} \overline{\alpha}$$

Example

$$\underline{\alpha b}\alpha c\alpha\overline{\alpha}\overline{d}\overline{\alpha}\overline{b}\overline{\alpha}\overline{b}\overline{\alpha}$$

Example

$$\underline{\alpha b}\alpha c\alpha\overline{\alpha}\overline{d}\overline{\alpha}\overline{b}\overline{\alpha}(\overline{b}\overline{\alpha})^i$$

Example

$$\underline{\alpha b \alpha c}\alpha\overline{\alpha}\overline{d}\overline{\alpha}\overline{b}\overline{\alpha}(\overline{b}\overline{\alpha})^i\overline{c}\overline{\alpha}\overline{b}\overline{\alpha}$$

Example

$$\alpha b\alpha c(\alpha b)^{i+1}\alpha d\alpha b\alpha c\alpha\bar{\alpha}\underline{\bar{d}\bar{\alpha}\bar{b}\bar{\alpha}(\bar{b}\bar{\alpha})^{i}\bar{c}\bar{\alpha}\bar{b}\bar{\alpha}}$$

Example

$$\alpha b \alpha c (\alpha b)^{i+1} \alpha d \alpha b \alpha c (\alpha b)^{i+1} \alpha d \alpha b \alpha c \alpha \bar{\alpha} \underline{\bar{d} \bar{\alpha} \bar{b} \bar{\alpha} (\bar{b} \bar{\alpha})^{i} \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}}$$

Example

$$\alpha b \alpha c (\alpha b)^{i+1} \alpha d \alpha b \alpha c (\alpha b)^{i+1} \alpha d \alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{b} \bar{\alpha} (\bar{b} \bar{\alpha})^{i} \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$$

Example

$$\alpha b \alpha c (\alpha b)^{i+1} \alpha d \alpha b \alpha c (\alpha b)^{i+1} \alpha d \underbrace{\alpha b \alpha c \alpha \bar{\alpha} \bar{d} \bar{\alpha} \bar{b} \bar{\alpha}}_{= z} (\bar{b} \bar{\alpha})^{i} \bar{c} \bar{\alpha} \bar{b} \bar{\alpha}$$

Proposition

$\mathcal{H}^*(z)$ *is not context-free.*

Questions

1.) Given $w \in \Sigma^*$, is $\mathcal{H}^*(w) \overset{?}{\in} \mathrm{REG}$ decidable?

2.) Does $w \in \Sigma^*$ exist s. t. $\mathcal{H}^*(w)$ is context-free but not regular?

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Non-crossing Words

w is non-crossing $\iff$ $\underbrace{\alpha \;\rule{2em}{0.4pt}\; \alpha \;\rule{0.5em}{0.4pt}}_{\overline{\alpha}\text{-free}}\overbrace{\overline{\alpha}\;\rule{2em}{0.4pt}\;\overline{\alpha}}^{\alpha\text{-free}} \rightarrow$

$z = \alpha b \alpha c \alpha \overline{\alpha} \overline{d} \overline{\alpha} \overline{b} \overline{\alpha}$ is non-crossing.

Theorem (Kari, Kopecki, Seki, 2012)

For non-crossing $w \in \Sigma^*$

1.) $\mathcal{H}^*(w) \overset{?}{\in} \mathrm{REG}$ *is decidable.*

2.) $\mathcal{H}^*(w)$ *is either regular or not context-free.*

α-prefixes and $\overline{\alpha}$-suffixes

$P_\alpha(w) = \{p_0, \dots, p_m\}$ s. t. $p_i\alpha$ is a prefix of w.
$S_{\overline{\alpha}}(w) = \{\overline{s_0}, \dots, \overline{s_n}\}$ s. t. $\overline{\alpha}\,\overline{s_i}$ is a suffix of w.

$$\mathcal{H}^*(w) = \{w\} \cup \bigcup_{\substack{p \in P_\alpha(w) \\ |p|+2k \le |w|}} \mathcal{H}^*(w\bar{p}) \cup \bigcup_{\substack{\bar{s} \in S_{\overline{\alpha}}(w) \\ |s|+2k \le |w|}} \mathcal{H}^*(sw)$$

Example

$z = \alpha b \alpha c \alpha \overline{\alpha} \bar{d} \overline{\alpha} \bar{b} \overline{\alpha}$
$P_\alpha(z) = \{\varepsilon, \alpha b, \alpha b \alpha c\}$
$S_{\overline{\alpha}}(z) = \{\varepsilon, \bar{b}\overline{\alpha}, \bar{d}\overline{\alpha}\bar{b}\overline{\alpha}\}$

$$\mathcal{H}^*(z) = \{z\} \cup \mathcal{H}^*(z\bar{b}\overline{\alpha}) \cup \mathcal{H}^*(z\bar{c}\overline{\alpha}\bar{b}\overline{\alpha}) \cup \mathcal{H}^*(\alpha b z) \cup \mathcal{H}^*(\alpha b \alpha d z)$$

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Non-regularity of $\mathcal{H}^*(z)$

Theorem (KKS, 2012)

For w in non-crossing with $p_1 = s_1$, $\mathcal{H}^*(w)$ is regular iff

1. $p_i \in \overline{S_{\overline{\alpha}}(w)}^*$ *or* $\overline{S_{\overline{\alpha}}(w)} \subseteq \{p_1, \ldots, p_i\}^*$ *for all* $i \leq m$ *and*
2. $s_i \in P_\alpha(w)^*$ *or* $P_\alpha(w) \subseteq \{s_1, \ldots, s_i\}^*$ *for all* $i \leq n$.

Example

$z = \alpha b \alpha c \alpha \overline{\alpha}\overline{d}\overline{\alpha}\overline{d}\overline{\alpha}$
$P_\alpha(z) = \{\varepsilon, \alpha b, \alpha b \alpha c\}$
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Example

$z = \alpha b \alpha c \alpha \overline{\alpha} \overline{d} \overline{\alpha} \overline{d} \overline{\alpha}$
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The Crossing Case

Questions

1.) Given $w \in \Sigma^*$, is $\mathcal{H}^*(w) \overset{?}{\in} \mathrm{REG}$ decidable?

2.) Does $w \in \Sigma^*$ exist s. t. $\mathcal{H}^*(w)$ is context-free but not regular?

3.) Find another "nice" class for which $\mathcal{H}^*(w) \overset{?}{\in} \mathrm{REG}$ is decidable.

4.) When does w occur only once in every $z \in \mathcal{H}^*(w)$?

5.) When can the original w in $z \in \mathcal{H}^*(w)$ be determined?

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Thank you!

Questions?