# On Singularities Of Extremal Periodic Strings

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# Outline



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- In1998 by *Fraenkel* and *Simpson* showed that the number of distinct squares in a string of length $n$ is at most $2n$ and hypothesized that the bound should be $n$.
  In 2005 *Ilie* provided a simpler proof and in 2007 presented an asymptotic upper bound of $2n - \Theta(\log n)$.
- In 1999 *Kolpakov* and *Kucherov* proved that the maximum number of runs in a string is linear in the string's length and conjectured that it is in fact bounded by the length.
  Many additional authors (*Rytter, Smyth, Simpson, Puglisi, Crochemore, Ilie, Kusano, Matsubara, Ishino, Bannai, Shinohara, FF*) contributed to improving the lower and upper bounds to the current asymptotic

  $$0.944565n \leq \rho(n) \leq 1.029n$$

We consider the role played by the size of the alphabet of the string in both problems and investigate the functions $\sigma_d(n)$ and $\rho_d(n)$, i.e. the maximum number of distinct primitively rooted squares, respectively runs, over all strings of length $n$ containing exactly $d$ distinct symbols. We revisit earlier results and conjectures and express them in terms of singularities of the two functions where a pair $(d, n)$ is a *singularity* if $\sigma_d(n) - \sigma_{d-1}(n-2) \geq 2$, or $\rho_d(n) - \rho_{d-1}(n-2) \geq 2$ respectively.

| $d$ \ $n-d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | **2** | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 7 | 8 | 9 | 10 | 11 | 12 |
| 3 | 2 | **3** | 3 | 4 | 4 | 5 | 6 | 7 | 8 | 8 | 9 | 10 | 11 | 12 |
| 4 | 2 | 3 | **4** | 4 | 5 | 5 | 6 | 7 | 8 | 9 | 9 | 10 | 11 | 12 |
| 5 | 2 | 3 | 4 | **5** | 5 | 6 | 6 | 7 | 8 | 9 | 10 | 10 | 11 | 12 |
| 6 | 2 | 3 | 4 | 5 | **6** | 6 | 7 | 7 | 8 | 9 | 10 | 11 | 11 | 12 |
| 7 | 2 | 3 | 4 | 5 | 6 | **7** | 7 | 8 | 8 | 9 | 10 | 11 | 12 | 12 |
| 8 | 2 | 3 | 4 | 5 | 6 | 7 | **8** | 8 | 9 | 9 | 10 | 11 | 12 | 13 |
| 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | **9** | 9 | 10 | 10 | 11 | 12 | 13 |
| 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **10** | 10 | 11 | 11 | 12 | 13 |
| 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | **11** | 11 | 12 | 12 | 13 |
| 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** | 12 | 13 | 13 |
| 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **13** | 13 | 14 |
| 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | **14** | 14 |
| 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | **15** |

Table 1.1: $(d, n-d)$ table for $\sigma_d(n)$ with $2 \le d \le 15$ and $2 \le n-d \le 15$

http://optlab.mcmaster.ca/~jiangm5/research/square.html

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$n-d$

| | | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | **2** | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 | ***8*** | 10 | 10 | 11 | 12 |
| | 3 | 2 | **3** | 3 | 4 | 5 | 6 | 6 | 7 | 8 | 9 | ***10*** | 11 | 11 | 12 |
| | 4 | 2 | 3 | **4** | 4 | 5 | 6 | 7 | 7 | 8 | 9 | 10 | 11 | 12 | 12 |
| | 5 | 2 | 3 | 4 | **5** | 5 | 6 | 7 | 8 | 8 | 9 | 10 | 11 | 12 | 13 |
| | 6 | 2 | 3 | 4 | 5 | **6** | 6 | 7 | 8 | 9 | 9 | 10 | 11 | 12 | 13 |
| | 7 | 2 | 3 | 4 | 5 | 6 | **7** | 7 | 8 | 9 | 9 | 10 | 11 | 112 | 13 |
| | 8 | 2 | 3 | 4 | 5 | 6 | 7 | **8** | 8 | 9 | 10 | 11 | 11 | 12 | 13 |
| $d$ | 9 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | **9** | 9 | 10 | 11 | 12 | 12 | 13 |
| | 10 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **10** | 10 | 11 | 12 | 13 | 13 |
| | 11 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | **11** | 11 | 12 | 13 | 13 |
| | 12 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | **12** | 12 | 13 | 14 |
| | 13 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | **13** | 13 | 14 |
| | 14 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | **14** | 14 |
| | 15 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | **15** |

Table 1.2: $(d, n-d)$ table for $\rho_d(n)$ with $2 \leq d \leq 15$ and $2 \leq n-d \leq 15$

`http://optlab.mcmaster.ca/~bakerar2/research/runmax/index.html`

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## Proposition

**($s_1$)** $0 \leq \sigma_d(n+1) - \sigma_d(n) \leq 2$ *for* $n \geq d \geq 2$

**($s_2$)** $\sigma_d(n) \leq \sigma_{d+1}(n+1)$ *for* $n \geq d \geq 2$

**($s_3$)** $\sigma_d(n) < \sigma_{d+1}(n+2)$ *for* $n \geq d \geq 2$

**($s_4$)** $\sigma_d(n) = \sigma_{d+1}(n+1)$ *for* $2d \geq n \geq d \geq 2$

**($s_5$)** $\sigma_d(n) \geq n-d$, $\sigma_d(2d+1) \geq d$ *and* $\sigma_d(2d+2) \geq d+1$ *for* $2d \geq n \geq d \geq 2$

**($s_6$)** $\sigma_{d-1}(2d-1) = \sigma_{d-2}(2d-2)$ *and* $0 \leq \sigma_d(2d) - \sigma_{d-1}(2d-1) \leq 1$ *for* $d \geq 4$

**($s_7$)** $1 \leq \sigma_{d+1}(2d+2) - \sigma_d(2d) \leq 2$ *for* $d \geq 2$.

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## Corollary

**($c_1$)** $\sigma_2(n) \leq 2n - 51$ *for* $n \geq 41$

**($c_2$)** $\sigma(n) \leq 2n - 19$ *for* $n \geq 30$.

## Conjecture

*For any* $n \geq d \geq 2$, $\sigma_d(n) \leq n - d$

## Theorem

*Let* $(d, 2d)$ *be the first singularity on the main diagonal, i.e. the least* $d$ *such that* $\sigma_d(2d) - \sigma_{d-1}(d-2) \geq 2$. *Then any square-maximal* $(d, 2d)$*-string does not contain a pair but must contain at least* $\lceil \frac{2d}{3} \rceil$ *singletons.*

## Theorem

**($e_1$)** *no* $(d, 2d)$ *singularity* $\iff \{\sigma_d(n) \le n{-}d$ *for* $n \ge d \ge 2\}$

**($e_2$)** $\{\sigma_d(n) \le n{-}d$ *for* $n \ge d \ge 2\} \iff$
$\{\sigma_d(4d) \le 3d$ *for* $d \ge 2\}$

**($e_3$)** $\{\sigma_d(n) \le n{-}d$ *for* $n \ge d \ge 2\} \iff$
$\{\sigma_d(2d{+}1){-}\sigma_d(2d) \le 1$ *for* $d \ge 2\}$

**($e_4$)** *no* $(d, 2d{+}1)$ *singularity* $\implies$
$\{$*no* $(d, 2d)$ *singularity and* $\sigma_d(n) \le n{-}d{-}1$ *for* $n > 2d \ge 4\}$

**($e_5$)** $\{\sigma_d(2d) = \sigma_d(2d{+}1)$ *for* $d \ge 2\} \implies$
$\{$*no* $(d, 2d)$ *singularity and* $\sigma_d(n) \le n{-}d{-}1$ *for* $n > 2d \ge 4\}$

**($e_6$)** $\{\sigma_d(2d) = \sigma_d(2d{+}1)$ *for* $d \ge 2\} \implies$
$\{$*square-maximal* $(d, 2d)$*-strings are, up to relabelling, unique and equal to* $a_1a_1a_2a_2a_2 \ldots a_da_d\}$

## Proposition

($r_1$) $\rho_d(n) \leq \rho_{d+1}(n{+}1)$ *for* $n \geq d \geq 2$

($r_2$) $\rho_d(n) \leq \rho_d(n{+}1)$ *for* $n \geq d \geq 2$

($r_3$) $\rho_d(n) < \rho_{d+1}(n{+}2)$ *for* $n \geq d \geq 2$

($r_4$) $\rho_d(n) = \rho_{d+1}(n{+}1)$ *for* $2d \geq n \geq d \geq 2$

($r_5$) $\rho_d(n) \geq n{-}d$, $\rho_d(2d{+}1) \geq d$ *and*
$\rho_d(2d{+}2) \geq d{+}1$ *for* $2d \geq n \geq d \geq 2$

($r_6$) $\rho_{d-1}(2d{-}1) = \rho_{d-2}(2d{-}2) = \rho_{d-3}(2d{-}3)$ *and*
$0 \leq \rho_d(2d){-}\rho_{d-1}(2d{-}1) \leq 1$ *for* $d \geq 5$

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## Proposition

*Let $(d, 2d)$ be the first singularity on the main diagonal, i.e. the least $d$ such that $\rho_d(2d) - \rho_{d-1}(2d-2) \geq 2$. Then any run-maximal $(d, 2d)$-string does not contain a symbol occurring exactly $2, 3, \ldots, 7$ or $8$ times, and must contains at least $\lceil \frac{7d}{8} \rceil$ singletons.*

## Conjecture

*For any $n \geq d \geq 2$, $\rho_d(n) \leq n - d$*

## Theorem

**($e_1$)** *no* $(d,2d)$ *singularity* $\iff \{\rho_d(n) \le n{-}d$ *for* $n \ge d \ge 2\}$

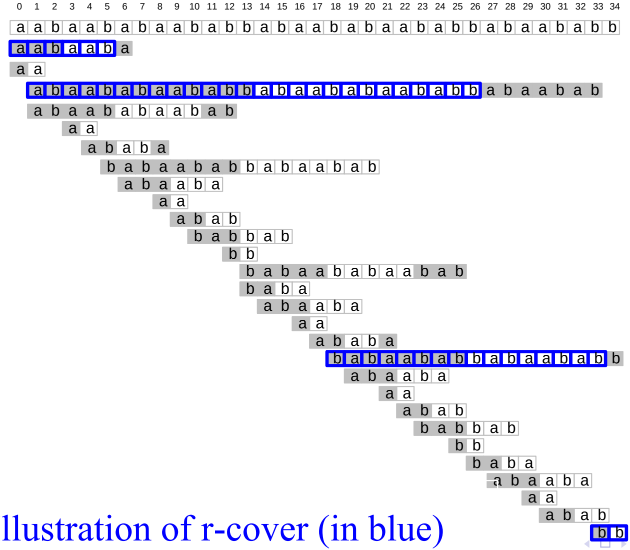
**($e_2$)** $\{\rho_d(n) \le n{-}d$ *for* $n \ge d \ge 2\} \iff$
$\{\rho_d(9d) \le 8d$ *for* $d \ge 2\}$

**($e_3$)** $\{\rho_d(n) \le n{-}d$ *for* $n \ge d \ge 2\} \iff$
$\{\rho_d(2d{+}1){-}\rho_d(2d) \le 1$ *for* $d \ge 2\}$

**($e_4$)** *no* $(d,2d{+}1)$ *singularity* $\Longrightarrow$
$\{$*no* $(d,2d)$ *singularity and* $\rho_d(n) \le n{-}d{-}1$ *for* $n > 2d \ge 4\}$

**($e_5$)** $\{\rho_d(2d) = \rho_d(2d{+}1)$ *for* $d \ge 2\} \Longrightarrow$
$\{$*no* $(d,2d)$ *singularity and* $\rho_d(n) \le n{-}d{-}1$ *for* $n > 2d \ge 4\}$

**($e_6$)** $\{\rho_d(2d) = \rho_d(2d{+}1)$ *for* $d \ge 2\} \Longrightarrow \{$*square-maximal* $(d,2d)$*-strings are, up to relabelling, unique and equal to* $a_1a_1a_2a_2a_2\dots a_da_d\}$




# Illustration of r-cover (in blue)

d = 2, n = 10 Covered: 154 not Covered: 357

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d = 2, n = 15 Covered: 4074 not Covered: 12,309

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d = 2, n = 20 Covered: 109,437 not Covered: 414,850

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d = 3, n = 10 Covered: 183 not Covered: 9,147

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d = 3, n = 15 Covered: 21,681 not Covered: 2,353,420

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d = 3, n = 20 Covered: 1,908,923 not Covered: 578,697,523

The values for $\sigma_d(n)$ and $\rho_d(n)$ computed to date:

- For $\sigma_d(n)$ function, so far we have found two singularities: $(3, 35)$ as $\sigma_3(35) = 25$ and $\sigma_2(33) = 23$, and $(3, 36)$ as $\sigma_3(36) = 26$ and $\sigma_2(34) = 24$.
- $\sigma_3(33) = 24 \geq \sigma_2(33) = 23$ (colorblueno binary string of length 33 attains the maximum)
- For $\rho_d(n)$ function, so far we have found three singularities: $(3, 15)$ as $\rho_3(15) = 10$ and $\rho_2(13) = 8$, $(3, 43)$ as $\rho_2(41) = 33$ and $\rho_3(43) = 35$, and $(4, 44)$, as $\rho_3(42) = 33$ and $\rho_4(44) \geq \rho_3(43) = 35$

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# *THANK YOU*