Are there patterns which are 6-avoidable but not 5-avoidable?

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April 14, 2013

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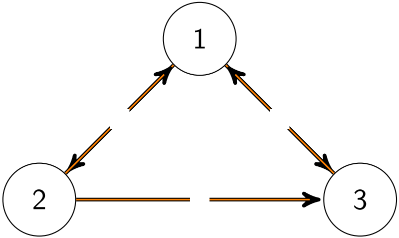
- Pattern p is k-**avoidable** if there are infinitely many words over $\{1, 2, \ldots, k\}$ which avoid p.
- The classic example is that xx is 3-avoidable but not 2-avoidable.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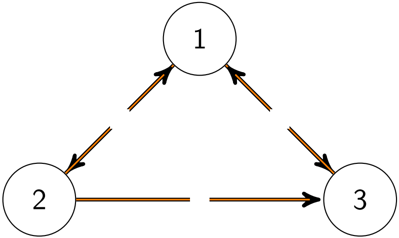

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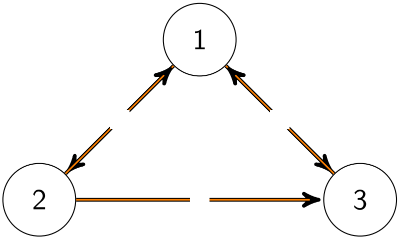

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- Any infinite squarefree words walkable on this graph must have final segments concatenated from 31, 312, 3121.

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- Theorem (1989 Baker, McNulty, Taylor) Pattern p is 4-avoidable but not 3-avoidable.

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- If an avoidable pattern p contains at most α distinct letters, then p is $(\alpha + 5)$-avoidable. (Mel'nichuk, 1988)