On the Brlek-Reutenauer conjecture

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Program

1 Defect and complexities

2 The Brlek-Reutenauer conjecture step by step

3 Proof of the Brlek-Reutenauer conjecture

- a finite word $w = w_0 w_1 \dots w_n$
- an infinite word $\mathbf{u} = u_0 u_1 u_2 \dots$

Factor and palindromic complexity

Let $\mathbf{u}$ be an infinite word

- **factor complexity** $\mathcal{C}_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$

$$\mathcal{C}_{\mathbf{u}}(n) = \text{the number of factors of length } n \text{ of } \mathbf{u}$$

- **palindromic complexity** $\mathcal{P}_{\mathbf{u}} : \mathbb{N} \to \mathbb{N}$

$$\mathcal{P}_{\mathbf{u}}(n) = \text{the number of palindromes of length } n \text{ contained in } \mathbf{u}$$

Theorem (Baláži, Masáková, Pelantová, 2007)

Let $\mathbf{u}$ be an infinite word with language closed under reversal, then

$$\mathcal{P}_{\mathbf{u}}(n) + \mathcal{P}_{\mathbf{u}}(n+1) \leq \mathcal{C}_{\mathbf{u}}(n+1) - \mathcal{C}_{\mathbf{u}}(n) + 2 \quad \textit{for all } n \in \mathbb{N}.$$

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Defect

Proposition (Droubay, Justin, Pirillo, 2001)

Every finite word w contains at most $|w| + 1$ palindromes (including the empty word).

Definition (Brlek, Hamel, Nivat, Reutenauer, 2004)

Let w be a finite word, then **defect** of w is

$$D(w) = |w| + 1 - \text{the number of palindromes contained in } w.$$

Let $\mathbf{u}$ be an infinite word, then **defect** of $\mathbf{u}$ is

$$D(\mathbf{u}) = \sup\{D(w) \mid w \text{ is a prefix of } \mathbf{u}\}.$$

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Relation of defect and complexities

Denote

$$T_{\mathbf{u}}(n) = \mathcal{C}_{\mathbf{u}}(n+1) - \mathcal{C}_{\mathbf{u}}(n) + 2 - \mathcal{P}_{\mathbf{u}}(n) - \mathcal{P}_{\mathbf{u}}(n+1).$$

Theorem (Bucci, De Luca, Glen, Zamboni, 2009)

Let $\mathbf{u}$ be an infinite word with language closed under reversal, then $D(\mathbf{u}) = 0$ if and only if $T_{\mathbf{u}}(n) = 0$ for all $n \in \mathbb{N}$.

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The Brlek-Reutenauer conjecture

Conjecture (Brlek, Reutenauer, 2011)

Let $\mathbf{u}$ *be an infinite word with language closed under reversal, then*

$$\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) = 2D(\mathbf{u}).$$

Proof done ibidem

- for finite words,
- for periodic words,
- for words with zero defect (using Theorem of Bucci et al.).

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The Brlek-Reutenauer conjecture

- Let w be a finite word, then

$$D(w) = \sum_{n=0}^{|w|} T_w(n).$$

- Since $D(w) \leq D(v)$ for any factor w of v,

$$\lim_{N\to\infty} D(u^{(N)}) = \sup_N D(u^{(N)}) = D(\mathbf{u}).$$

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- Therefore

$$D(\mathbf{u}) = \lim_{N\to\infty} \sum_{n=0}^{N} T_{u^{(N)}}(n) \stackrel{?}{=} \lim_{N\to\infty} \sum_{n=0}^{N} T_{\mathbf{u}}(n).$$

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Difference between sums

- The following two expressions are not equal

$$\sum_{n=0}^{N} T_{u^{(N)}}(n) = \sum_{n=0}^{N} \mathcal{C}_{u^{(N)}}(n+1)-\mathcal{C}_{u^{(N)}}(n)+2-\mathcal{P}_{u^{(N)}}(n)-\mathcal{P}_{u^{(N)}}(n+1)$$

$$\sum_{n=0}^{N} T_{\mathbf{u}}(n) = \sum_{n=0}^{N} \mathcal{C}_{\mathbf{u}}(n+1) - \mathcal{C}_{\mathbf{u}}(n) + 2 - \mathcal{P}_{\mathbf{u}}(n) - \mathcal{P}_{\mathbf{u}}(n+1)$$

- Consider $\mathbf{u} = (0120210)^{\omega}$, then

$$\sum_{n=0}^{3} T_{u^{(3)}}(n) = 0 = 2D(u^{(3)}), \quad \text{but} \sum_{n=0}^{3} T_{\mathbf{u}}(n) = 2.$$

The Brlek-Reutenauer conjecture - partial proof

Theorem (Balková, Pelantová, Starosta, 2011)

Let $\mathbf{u}$ *be a uniformly recurrent infinite word with language closed under reversal, then the Brlek-Reutenauer conjecture holds for* $\mathbf{u}$*.*

Proof.

Construction of a periodic word $\mathbf{v}$ with language closed under reversal such that $D(\mathbf{v}) = D(\mathbf{u})$ and $T_{\mathbf{v}}(n) = T_{\mathbf{u}}(n)$. Use of validity of the BR conjecture for periodic words.

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Lemma (essential property of uniformly recurrent words)

Let $\mathbf{u}$ *be a uniformly recurrent infinite word with finite defect. Then the set* $\{w \mid ww \text{ is a factor of } \mathbf{u}\}$ *is infinite.*

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The Brlek-Reutenauer conjecture - complete proof

Theorem (Balková, Pelantová, Starosta, 2012)

The Brlek-Reutenauer conjecture holds, i.e., $\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) = 2D(\mathbf{u})$ for every infinite word $\mathbf{u}$ with language closed under reversal.

Proof.

2 steps:

1. If $D(\mathbf{u}) < \infty$ and $\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) < \infty$, then $\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) = 2D(\mathbf{u})$.
2. $D(\mathbf{u}) < \infty$ if and only if $\sum_{n=0}^{\infty} T_{\mathbf{u}}(n) < \infty$.

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Proof.

Find H such that

1. $D(\mathbf{u}) = D(v)$ for every prefix v of $\mathbf{u}$ of length $\geq H - 1$,
2. $T_{\mathbf{u}}(n) = 0$ for all $n \geq H$.

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Take any prefix p of $\mathbf{u}$ containing all factors of length $\leq H$. Then $2D(\mathbf{u}) = 2D(p) = \sum_{n=0}^{|p|} T_p(n) = \sum_{n=0}^{H-1} T_p(n) + \sum_{n=H}^{|p|} T_p(n) = \sum_{n=0}^{H-1} T_{\mathbf{u}}(n) = \sum_{n=0}^{\infty} T_{\mathbf{u}}(n)$.

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□

Thank you for attention!

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