Solving Triangular Systems

A triangular matrix has zeros either above the diagonal, or below the diagonal.

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}
\text{is...}
\begin{bmatrix}
a & 0 & 0 & 0 \\
b & c & 0 & 0 \\
d & e & f & 0 \\
g & h & i & j
\end{bmatrix}
\text{is...}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\text{is...}
\]

Triangular matrices occur in certain matrix factorizations, and are a useful type of matrix.

Solving Upper-Triangular Systems: Back Substitution

\[
Ux = z \quad U \text{ is upper-triangular matrix}
\]

\[
\begin{bmatrix}
U_{11} & U_{12} & \cdots & U_{1N} \\
0 & U_{22} & \cdots & U_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & U_{N-1,N-1} & U_{N,N}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{bmatrix} =
\begin{bmatrix}
z_1 \\
z_2 \\
\vdots \\
z_N
\end{bmatrix}
\]

Start with the last row

\[
U_{N,N} \cdot x_N = z_N \quad \Rightarrow \quad x_N = \frac{z_N}{U_{N,N}}
\]

Now it's easy to solve for \( x_{n-1} \) in the second-last row.
The $i$-th row...

$$U_{n-1,n-1} x_{n-1} + u_{n-1,n} x_n = z_{n-1}$$

$$\Rightarrow x_{n-1} = \frac{z_{n-1} - u_{n-1,n} x_n}{U_{n-1,n-1}}$$

$$U_i x_i + U_{i,i+1} x_{i+1} + \cdots + u_{i,n} x_n = z_i$$

$$\Rightarrow x_i = \frac{z_i - (u_{i,i+1} x_{i+1} + \cdots + u_{i,n} x_n)}{u_{ii}}$$

$$x_i = \frac{z_i - \sum_{j=i+1}^{N} u_{ij} x_j}{u_{ii}}$$

**Back Substitution Algorithm** (a.k.a. Back Solve)
(see page 101 in the course notes)

**Complexity**
For each $i$, the $j$-loop performs $2(N-i)$ flops (floating-point operations). Together with $\div u_{ii}$

$$\Rightarrow \text{flops} = 2(N-i) + 1$$

**Sum over i**

$$\text{Total flops} = \sum_{i=1}^{N} \left( 2(N-i) + 1 \right) = \sum_{i=1}^{N} 2N - 2i + 1$$

$$= 2N^2 + N - 2 \sum_{i=1}^{N} i$$

$$= 2N^2 + N - \frac{N(N+1)}{2}$$
\[ Lx = z \quad L \text{ is } N \times N \text{ lower-} \Delta \]

\[
\begin{bmatrix}
 l_{11} & 0 & \cdots \\
 l_{12} & l_{22} & 0 \\
 \vdots & \ddots & \ddots \\
 l_{1N} & l_{2N} & \cdots & l_{NN}
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_N
\end{bmatrix}
= \begin{bmatrix}
 z_1 \\
 z_2 \\
 \vdots \\
 z_N
\end{bmatrix}
\]

From 1st row,
\[ x_1 = \frac{z_1}{l_{11}} \]

For \( i \text{th} \) row,
\[ l_{i1} x_1 + l_{i2} x_2 + \cdots + l_{i,i-1} x_{i-1} + l_{ii} x_i = z_i \]

\[ x_i = \frac{z_i - \left( l_{i1} x_1 + \cdots + l_{i,i-1} x_{i-1} \right)}{l_{ii}} \]

\[ x_i = \frac{z_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}} \]
Gaussian Elimination

To solve a system of linear equations

\[ A\mathbf{x} = \mathbf{b} \quad \text{e.g.} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix} \]

one can use Gaussian elimination.

1) Form the augmented matrix.

\[
\begin{bmatrix}
1 & 1 & 2 & | & 8 \\
-1 & -2 & 3 & | & 1 \\
3 & -7 & 4 & | & 10
\end{bmatrix}
\]

2) Perform linear row operations to get an upper-Δ form

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 2 & | & 8 \\
0 & -1 & 5 & | & 9 \\
3 & -7 & 4 & | & 10
\end{bmatrix}
& \xrightarrow{\text{\(\Theta + 1\)}}
\begin{bmatrix}
1 & 2 & 0 & | & 8 \\
0 & -1 & 5 & | & 9 \\
0 & -4 & -2 & | & -14
\end{bmatrix}
\quad \text{\(\Theta - 3\Theta\)}
\begin{bmatrix}
1 & 2 & 0 & | & 8 \\
0 & -1 & 5 & | & 9 \\
0 & 0 & -1 & | & -12
\end{bmatrix}
\quad \text{\(\Theta - 5\Theta\)}
\begin{bmatrix}
1 & 2 & 0 & | & 8 \\
0 & -1 & 5 & | & 9 \\
0 & 0 & 1 & | & 2
\end{bmatrix}
\end{align*}
\]

3) You might have been taught to follow this with more row operations to get a diagonal matrix.

\[
\begin{align*}
\begin{bmatrix}
1 & 2 & 0 & | & 8 \\
0 & -1 & 5 & | & 9 \\
0 & 0 & 1 & | & 2
\end{bmatrix}
& \xrightarrow{\text{\(\Theta - 1\Theta\)}}
\begin{bmatrix}
1 & 0 & 0 & | & 3 \\
0 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 2
\end{bmatrix}
\end{align*}
\]

A better way is to use back substitution after step 2. Solve

\[
\begin{bmatrix}
1 & 2 & \ | & x_1 \\
0 & -5 & \ | & x_2 \\
0 & 1 & \ | & x_3
\end{bmatrix}
= \begin{bmatrix} 8 \\ -9 \\ 2 \end{bmatrix}
\]
LU Factorization
Any square matrix $A$ can be factored into a product of an upper-triangular and lower-triangular matrices such that

$$LU = PA$$

$P$ is a permutation matrix used to swap rows. LU factorization is essentially the same as Gaussian elimination.

In this case, there is no way to use the pivot element to get rid of the 5. So, swap rows 2 and 3.

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 6 & -1 \\ -2 & 2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

We will go over the details later, but for now I'll simply state that LU factorization takes $O(N^3)$ flops.

Applications of LU Factorization
a) Solving $Ax = b$

$$Ax = b \Rightarrow PAx = Pb \Rightarrow LUx = Pb$$
Two steps to compute \( \boldsymbol{x} \)

1) Solve \( L \boldsymbol{z} = \boldsymbol{Pb} \) for \( \boldsymbol{z} \) \( (N^2) \)

2) Solve \( U \boldsymbol{x} = \boldsymbol{z} \) for \( \boldsymbol{x} \) \( (N^2) \)

Gaussian elimination = \( \text{LU factorization} + \text{forward sub} \) + \( \text{back sub} \)

\[ \mathcal{O}(N^3) \quad \mathcal{O}(N^2) \]

b) Solving \( \boldsymbol{A} \boldsymbol{x} = \boldsymbol{b} \)

Suppose \( \boldsymbol{x} \) and \( \boldsymbol{b} \) each have \( M \) columns.

\[
\boldsymbol{A} \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_M \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_M \end{bmatrix}
\]

1) Factor: \( \boldsymbol{LU} = \boldsymbol{PA} \) \( \mathcal{O}(N^3) \)

2) Solve: \( \boldsymbol{LU} \boldsymbol{x}_i = \boldsymbol{Pb}_i \) \( \mathcal{O}(N^2) \) each

\[
\begin{aligned}
    \boldsymbol{L} \boldsymbol{U} \boldsymbol{x}_1 &= \boldsymbol{Pb}_1 \\
    \vdots \\
    \boldsymbol{L} \boldsymbol{U} \boldsymbol{x}_M &= \boldsymbol{Pb}_M
\end{aligned}
\]

Take-home message: Do the expensive LU factorization once, and use it for each of the \( M \) systems.

Gaussian Elimination (GE)

\( 2 \times 2 \) example

\[
\begin{aligned}
    a_{11} \boldsymbol{x}_1 + a_{12} \boldsymbol{x}_2 &= b_1 \\
    a_{21} \boldsymbol{x}_1 + a_{22} \boldsymbol{x}_2 &= b_2
\end{aligned}
\]

\[ \frac{a_{21}}{a_{11}} \frac{\text{row 2} - \text{row 1}}{\text{row 1}} \Rightarrow \left( a_{21} - \frac{a_{21}}{a_{11}} a_{11} \right) \boldsymbol{x}_1 + \left( a_{22} - \frac{a_{21}}{a_{11}} a_{12} \right) \boldsymbol{x}_2 \]
In matrix form:

\[
\begin{bmatrix}
a_{ii} & a_{i2} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

For general \( N \), the big picture of GE...

GE Version 1:

for \( i = 1: \text{N-1} \)

eliminate \( x_i \) from rows \( i+1 \) to \( N \)

end

At the \( i \)-th stage of GE...

\[
\begin{bmatrix}
x & x & x & x \\
0 & a_{ii} & x & x \\
0 & 0 & a_{kk} & x & x \\
0 & 0 & 0 & a_{kk}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
x & x & x & x \\
0 & a_{ii} & x & x \\
0 & 0 & x & x \\
0 & 0 & 0 & x
\end{bmatrix}
\rightarrow
\begin{bmatrix}
x & x & x & x \\
0 & 0 & 0 & x \\
0 & 0 & 0 & x \\
0 & 0 & 0 & x
\end{bmatrix}
\]
GE Version 2:
for i = 1:N-1
  for k = i+1:N
    mult = \( \frac{a_{ki}}{a_{ii}} \)
    row(k) = row(k) - mult*row(i)
  end
end

To update the entire row \( k \):

\[
\begin{bmatrix}
0 & \cdots & 0 & a_{ii} & \cdots & 0 \\
0 & \cdots & 0 & a_{kj} & \cdots & 0
\end{bmatrix}
\]

pivot row \( i \)
current row \( k \)

for j = i+1:N

\( a_{kj} = a_{kj} - mult*a_{ij} \)
end

(Note: \( a_{ki} = 0 \) by design)

Final GE Algorithm:
for i = 1:N-1
  for k = i+1:N

Linear Algebra Page 23
\[ \text{mult} = \frac{a_{ki}}{a_{ii}} \]

\begin{verbatim}
for j = i+1:N
    a_{kj} = a_{kj} - \text{mult} \cdot a_{ij}
end
\end{verbatim}

\[ a_{ki} = 0 \]

end

end

Notes:
1) The lower-triangular part is all 0.
2) We may use those elements to store the multipliers.
LU Factorization Algorithm

Consider the first step of GE,

\[ A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11}^{(1)} & \cdots & a_{1N}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & a_{NN}^{(1)} \end{bmatrix} = A^{(1)} \]

Matrix interpretation:
row operations ↔ matrix multiplication

i.e., \( M^{(1)} A = A^{(1)} \)

where

\[ M^{(1)} = \begin{bmatrix} 1 & & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & & \\ & \ddots & \ddots & & \\ & & -\frac{a_{N1}}{a_{11}} & 1 & \\ & & & & 1 \end{bmatrix} \]

In general, at the k-th step...

\( M^{(k)} A^{(k-1)} = A^{(k)} \)

where

\[ M^{(k)} = \begin{bmatrix} 1 & & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & & \\ & \ddots & \ddots & & \\ & & -\frac{a_{k1}}{a_{11}} & 1 & \\ & & & \ddots & \ddots \\ & & & & 1 \end{bmatrix} \]
The effect of left-multiplying $A^{(k-1)}$ by $M^{(k)}$ is to eliminate $x_k$ from rows $k+1$ to $N$.

At the final step: $A^{(N-1)}$ is upper-$\Delta$.

Recall that $M^{(k)} A^{(k-1)} = A^{(k)}$

For $k = 1 \Rightarrow M^{(1)} A = A^{(1)}$

$$M^{(2)}(M^{(1)} A) = M^{(2)} A^{(1)} = A^{(2)}$$

$$\vdots$$

$$M^{(N-1)} \cdots M^{(1)} A = A^{(N-1)} = U$$

$$\Rightarrow A = [M^{(N-1)} \cdots M^{(1)}]^{-1} U$$

**Amazing Facts**

1) If $B$ and $C$ are lower-$\Delta$ and unit diagonal, then so is $BC$.
2) If $B$ is lower-$\Delta$ and unit diagonal, then so is $B^{-1}$.

By fact (1), $M^{(N-1)} \cdots M^{(1)}$ is lower-$\Delta$ unit-diag.

By fact (2), $[M^{(N-1)} \cdots M^{(1)}]^{-1}$ is lower-$\Delta$ unit-diag.
By fact (2), \([M^{(n-1)} \ldots M^{(1)}]^{-1}\) is lower-\(\Delta\) unit-diagonal.

Define \(L = [M^{(n-1)} \ldots M^{(1)}]^{-1}\)

Theorem: \(A = LU\)

\(L\) is lower-\(\Delta\) and unit diagonal
\(U\) is upper-\(\Delta\).

Properties of \(M^{(n)}\):

1) \((M^{(k)})^{-1} = \begin{bmatrix}
\text{diag}(a_{kk}) & \frac{a_{k+1,k}}{a_{kk}} & \frac{a_{k+2,k}}{a_{kk}} & \vdots & \frac{a_{nk}}{a_{kk}} \\
0 & 0 & \ddots & \vdots & 0 \\
0 & 0 & \ddots & \vdots & 0 \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \vdots & 0 & 1
\end{bmatrix}\)

2) \(L = [M^{(n-1)} \ldots M^{(1)}]^{-1} = (M^{(1)})^{-1} \ldots (M^{(n-1)})^{-1}\)

\(L = \begin{bmatrix}
\text{diag}(a_{kk}) & \frac{a_{k+1,k}}{a_{kk}} & \frac{a_{k+2,k}}{a_{kk}} & \vdots & \frac{a_{nk}}{a_{kk}} \\
0 & 0 & \ddots & \vdots & 0 \\
0 & 0 & \ddots & \vdots & 0 \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \vdots & 0 & 1
\end{bmatrix}\)

i.e. the \(k\)th col. of \(L\) is the \(k\)th col. of \((M^{(1)})^{-1}\).

\(\text{multipliers}\)
e.g. \[
\begin{bmatrix}
1 & 0 & 0 \\
\alpha & 1 & 0 \\
\beta & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
\alpha & 1 & 0 \\
\beta & 1 & 1
\end{bmatrix}
\]

Note that \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\beta & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\beta & 0 & 1
\end{bmatrix}
\]
is not so straightforward.

More precisely, write \(L\) as

\[
L_{jk} = \begin{cases} 
1 & \text{if } j = k \\
0 & \text{if } j < k \\
\frac{a_{jk}}{\lambda_k} & \text{if } j > k 
\end{cases}
\]

\(j = 1, \ldots, N\)

Example:

\[
A = \begin{bmatrix}
2 & -1 & 3 \\
-4 & 6 & -5 \\
6 & 12 & 14
\end{bmatrix}
\rightarrow \begin{bmatrix}
2 & -1 & 3 \\
0 & 4 & 1 \\
0 & 16 & 7
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
3 & 4 & 1
\end{bmatrix}
\rightarrow \begin{bmatrix}
2 & -1 & 3 \\
0 & 4 & 1 \\
0 & 0 & 3
\end{bmatrix} = U
\]

Check:

\[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
3 & 4 & 1
\end{bmatrix}
\begin{bmatrix}
2 & -1 & 3 \\
0 & 4 & 1 \\
0 & 0 & 3
\end{bmatrix}
= \begin{bmatrix}
2 & -1 & 3 \\
-4 & 6 & -5 \\
0 & 13 & 16
\end{bmatrix}
\]
Stability of LU Factorization

In LU factorization, a problem arises when we have:

1. A zero pivot, i.e., \( a_{kk}^{(k-1)} = 0 \)
   - multipliers \( \frac{a_{ik}}{a_{kk}} \) are undefined

2. \( a_{kk}^{(k-1)} \neq 0 \)
   - multipliers become large
   - calculations become unstable

Pivoting

Find

\[
\max_{k \leq j \leq N} |a_{jk}^{(k-1)}| = |a_{k^*k}^{(k-1)}|
\]

Swap rows \( k^* \) and \( k \), and continue.

Note:

\( a_{k^*k}^{(k-1)} \neq 0 \)

Otherwise, \( a_{jk}^{(k-1)} = 0 \) \( \forall k \neq j \in N \)

\( \Rightarrow A \) is singular
As mentioned earlier, these row-swapping operations can be represented by matrix multiplication by a permutation matrix, \( P \). For example, to swap rows \( i \) & \( j \), simply swap rows \( i \) & \( j \) in the identity matrix.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = P
\]

Let's put it all together.

\[
\begin{align*}
A &= \begin{bmatrix}
1 & 1 & 1 \\
4 & 16 & 64 \\
2 & 2 & 8 \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
4 & 16 & 64 \\
1 & 1 & 1 \\
2 & 2 & 8 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{align*}
2 - \frac{1}{4} & \begin{bmatrix}
4 & 16 & 64 \\
0 & -3 & -15 \\
0 & -6 & -24 \\
\end{bmatrix} \\
\begin{bmatrix}
\frac{1}{4} & 1 & 0 \\
\frac{1}{2} & 1 \\
1 & 1 & 1.4 \\
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]
\[ \begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & -3 & -15 \end{bmatrix} \]

\[ P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix} \]

\[ L \]

\[ \begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & 0 & -3 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 2 & 2 & 8 \end{bmatrix} \]

\[ LU = PA \]

**Example:**

\[ A = \begin{bmatrix} -6 & 27 & -42 & -15 \\ -24 & 12 & 24 & -12 \\ -12 & 14 & -10 & 20 \\ -6 & -9 & 42 & 15 \end{bmatrix} \]

\[ \begin{bmatrix} \text{Swap 1 & 2} \end{bmatrix} \]

\[ \begin{bmatrix} -24 & 12 & 24 & -12 \end{bmatrix} \]

\[ P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]
\[
\begin{bmatrix}
-24 & 12 & 24 & -12 \\
-6 & 27 & -42 & -15 \\
-12 & 14 & -10 & 20 \\
-6 & -9 & 42 & 15 \\
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\text{②} - \frac{1}{4} \text{①} \quad \begin{bmatrix}
-24 & 12 & 24 & -12 \\
\frac{1}{4} & 24 & -48 & -12 \\
\frac{1}{2} & 8 & -22 & 26 \\
\frac{1}{4} & -12 & 36 & 18 \\
\end{bmatrix}
\]

\[
\text{③} - \frac{1}{3} \text{②} \quad \begin{bmatrix}
-24 & 12 & 24 & -12 \\
\frac{1}{4} & 24 & -48 & -12 \\
\frac{1}{2} & \frac{1}{3} & -6 & 30 \\
\frac{1}{4} & -\frac{1}{2} & 12 & 12 \\
\end{bmatrix}
\]

\[
\text{Swap ③ & ④} \quad \begin{bmatrix}
-24 & 12 & 24 & -12 \\
\frac{1}{4} & 24 & -48 & -12 \\
\frac{1}{2} & \frac{1}{3} & -6 & 30 \\
\frac{1}{4} & -\frac{1}{2} & 12 & 12 \\
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-24 & 12 & 24 & -12 \\
\frac{1}{4} & -24 & -48 & -12 \\
\frac{1}{4} & -\frac{1}{2} & 12 & 12 \\
\frac{1}{2} & \frac{1}{3} & -\frac{1}{2} & 36
\end{bmatrix}
\]

\[
\therefore \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-6 & 27 & -42 & -15 \\
-24 & 12 & 24 & -12 \\
-12 & 14 & -10 & 20 \\
-6 & -9 & 42 & 15
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{4} & 1 & 0 & 0 \\
\frac{1}{4} & -\frac{1}{2} & 1 & 0 \\
\frac{1}{2} & \frac{1}{3} & -\frac{1}{2} & 1
\end{bmatrix}
\begin{bmatrix}
-24 & 12 & 24 & -12 \\
0 & 24 & -48 & -12 \\
0 & 0 & 12 & 12 \\
0 & 0 & 0 & 36
\end{bmatrix}
\]