Goal: To learn how to efficiently solve triangular linear systems.

Solving Triangular Systems

A triangular matrix has zeros either above the diagonal, or below the diagonal.

eg. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is ... $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix}$ is...

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is...

Triangular matrices occur in certain matrix factorizations, and are a useful type of matrix.

Solving Upper-Triangular Systems: Back Substitution

$Ux = z$ U is upper-Δ matrix

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & & u_{1N} \\ 0 & u_{22} & \cdots & & u_{2N} \\ \vdots & & \ddots & & \\ \vdots & & & u_{N-1,N-1} & u_{N-1,N} \\ 0 & \cdots & & 0 & u_{N,N} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_N \end{bmatrix}$$

Start with the last row

$$u_{N,N}\, x_N = z_N \implies x_N = \frac{z_N}{u_{N,N}}$$

Now it's easy to solve for x_{N-1} in the second-last row.

$$u_{N-1,N-1}\, x_{N-1} + u_{N-1,N}\, x_N = z_{N-1}$$

$$\Rightarrow \quad x_{N-1} = \frac{z_{N-1} - u_{N-1,N}\, x_N}{u_{N-1,N-1}}$$

The i-th row...

$$u_{ii}\, x_i + u_{i,i+1}\, x_{i+1} + \cdots + u_{iN}\, x_N = z_i$$

$$\Rightarrow x_i = \frac{z_i - (u_{i,i+1} x_{i+1} + \cdots + u_{iN} x_N)}{u_{ii}}$$

$$x_i = \frac{z_i - \sum_{j=i+1}^{N} u_{ij} x_j}{u_{ii}}$$

<u>Back Substitution Algorithm</u> (a.k.a. Back Solve)
(see page 101 in the course notes)

<u>Complexity</u>
For each i, the j-loop performs $2(N-i)$ flops (floating-point operations). Together with $\div\, u_{ii}$ $\Rightarrow$ $\text{flops} = 2(N-i)+1$

Sum over i

$$\text{Total flops} = \sum_{i=1}^{N} \big(2(N-i)+1\big) = \sum_{i=1}^{N} 2N - 2i + 1$$

$$= 2N^2 + N - 2\sum_{i=1}^{N} i$$

$$= 2N^2 + N - \frac{\cancel{2}\, N(N+1)}{\cancel{2}}$$

$$= 2N^2 + N - \frac{\cancel{2}N(N+1)}{\cancel{2}}$$

$$= 2N^2 + \cancel{N} - N^2 - \cancel{N}$$

$$= \boxed{N^2}$$

Forward Substitution

$Lx = z$ $\quad$ L is $N \times N$ lower-Δ

$$\begin{bmatrix} l_{11} & 0 & \cdots & \\ l_{21} & l_{22} & 0 & \cdots \\ \vdots & & & \\ l_{N1} & l_{N2} & \cdots & l_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$

From 1^{st} row,

$$\Rightarrow x_1 = \frac{z_1}{l_{11}}$$

ith row

$$l_{i1}x_1 + l_{i2}x_2 + \cdots + l_{i,i-1}x_{i-1} + l_{ii}x_i = z_i$$

$$\Rightarrow x_i = \frac{z_i - (l_{i1}x_1 + \cdots + l_{i,i-1}x_{i-1})}{l_{ii}}$$

$$x_i = \frac{z_i - \sum_{j=1}^{i-1} l_{ij}x_j}{l_{ii}}$$

Gaussian Elimination

To solve a system of linear equations

$$Ax = b \qquad \text{e.g.} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

one can use Gaussian elimination.

1) Form the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

2) Perform linear row operations to get an upper-Δ form

$$\begin{array}{c} \\ ②+① \\ ③-3① \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] \Rightarrow \begin{array}{c} \\ \\ ③-10② \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right] \Rightarrow \begin{array}{c} \\ -② \\ \frac{-1}{52}③ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

3) You might have been taught to follow this with more row operations to get a diagonal matrix.

$$\Rightarrow \cdots \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \therefore \text{solution is} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

A better way is to use back substitution after step 2.

Solve

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 2 \end{bmatrix}$$

Goal: To review how row operations can be used to reduce and solve a linear system.

LU Factorization

Any square matrix A can be factored into a product of an upper-triangular and lower-triangular matrices such that

$$LU = PA$$

P is a permutation matrix used to swap rows.

LU factorization is essentially the same as Gaussian elimination.

pivot element

new pivot element

eg. $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 6 & -1 \\ -2 & 2 & -3 \end{bmatrix} \Rightarrow \begin{matrix} ② - \frac{4}{2}① \\ ③ + \frac{+2}{2}① \end{matrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{bmatrix} \Rightarrow \ldots$

In this case, there is no way to use the pivot element to get rid of the 5. So, swap rows 2 and 3.

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{permutation matrix}} \begin{bmatrix} 2 & 3 & -1 \\ 0 & 0 & 1 \\ 0 & 5 & -4 \end{bmatrix}$$

We will go over the details later, but for now I'll simply state that LU factorization takes $\mathcal{O}(N^3)$ flops.

Applications of LU Factorization

a) Solving $Ax = b$

$$Ax = b \Rightarrow PAx = Pb \Rightarrow L\underbrace{Ux}_{z} = Pb$$

Two steps to compute x

1) Solve $Lz = Pb$ for z (N^2)
2) Solve $Ux = z$ for x (N^2)

Gaussian elimination = $\underbrace{\boxed{\text{LU factorization}}}_{O(N^3)}$ + $\underbrace{\boxed{\text{forward sub + back sub}}}_{O(N^2)}$

b) Solving $AX = B$

Suppose X and B each have M columns.

$$A\left[x_1 \middle| \cdots \middle| x_M\right] = \left[b_1 \middle| \cdots \middle| b_M\right]$$

1) Factor: $LU = PA$ $\quad O(N^3)$
2) Solve: $\left.\begin{array}{l} LUx_1 = Pb_1 \\ \vdots \\ LUx_M = Pb_M \end{array}\right\} O(N^2)$ each

Take-home message: Do the expensive LU factorization once, and use it for each of the M systems.

Gaussian Elimination (GE)

2×2 example

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 & (1) \\ a_{21}x_1 + a_{22}x_2 = b_2 & (2) \end{cases}$$

$$(2) - \frac{a_{21}}{a_{11}}(1) \Rightarrow \left(a_{21} - \frac{a_{21}}{a_{11}}a_{11}\right)x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2$$

$$② - \frac{a_{21}}{a_{11}} ① \Rightarrow \underbrace{\left(a_{21} - \frac{a_{21}}{a_{11}} a_{11}\right)}_{0} x_1 + \underbrace{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)}_{a_{22}^{(1)}} x_2 = \underbrace{b_2 - \frac{a_{21}}{a_{11}} b_1}_{b_2^{(1)}}$$

In matrix form: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$② - \frac{a_{21}}{a_{11}} ① \Rightarrow \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ 0 & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}$$

For general N , the big picture of GE...

$$\underset{A}{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}} \to \underset{A^{(1)}}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}} \to \underset{A^{(2)}}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}} \to \underset{A^{(3)}}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}}$$

GE Version 1:

for i = 1:N-1

eliminate x_i from rows i+1 to N

end

At the i-th stage of GE...

$$\begin{array}{r} \\ \text{pivot row } i \rightarrow \\ \\ \text{current row } k \rightarrow \end{array} \begin{bmatrix} x & x & x & x \\ 0 & a_{ii} & x & x \\ 0 & 0 & x & x \\ 0 & a_{ki} & x & x \end{bmatrix} \xrightarrow{\quad ⓚ - \frac{a_{ki}}{a_{ii}} ⓘ \quad} \begin{bmatrix} x & x & x & x \\ 0 & a_{ii} & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$k = i+1, \ldots, N$

GE Version 2:

for i = 1:N-1

for k = i+1:N

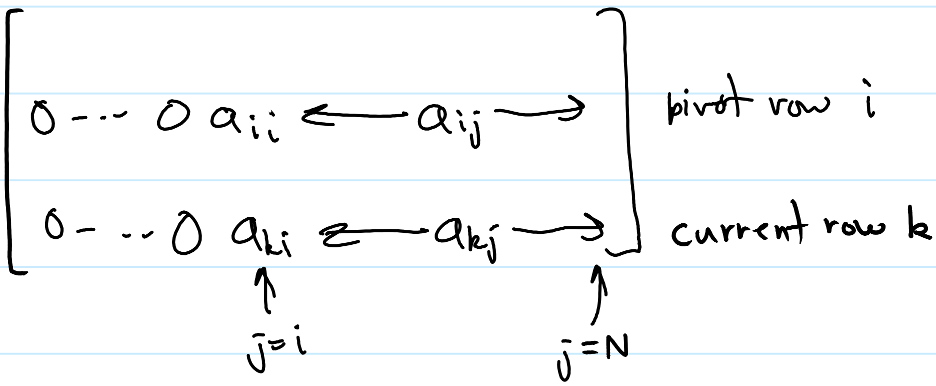
mult = a_{ki}/a_{ii}

row(k) = row(k) - mult*row(i)

end

end

To update the entire row k:



for j = i+1:N

$a_{kj} = a_{kj}$ - mult*a_{ij}

end

(Note: $a_{ki} = 0$ by design)

Final GE Algorithm:

for i = 1:N-1

for k = i+1:N

```
            mult = a_ki/a_ii
            for j = i+1:N
                a_kj = a_kj - mult*a_ij
            end
            a_ki = 0
        end

    end
```

Notes: 1) The lower-triangular part is all 0.
2) We may use those elements to store the multipliers.

Goal: To find out how to compute LU factorization.

LU Factorization Algorithm

Consider the first step of GE,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \xrightarrow[\text{operations}]{\text{row}} \begin{bmatrix} a_{11}^{(1)} & & & a_{1N}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & \\ \vdots & \vdots & & \\ 0 & a_{N2}^{(1)} & \cdots & a_{NN}^{(1)} \end{bmatrix} = A^{(1)}$$

Matrix interpretation:

row operations $\Leftrightarrow$ matrix multiplication

i.e. $M^{(1)} A = A^{(1)}$

where

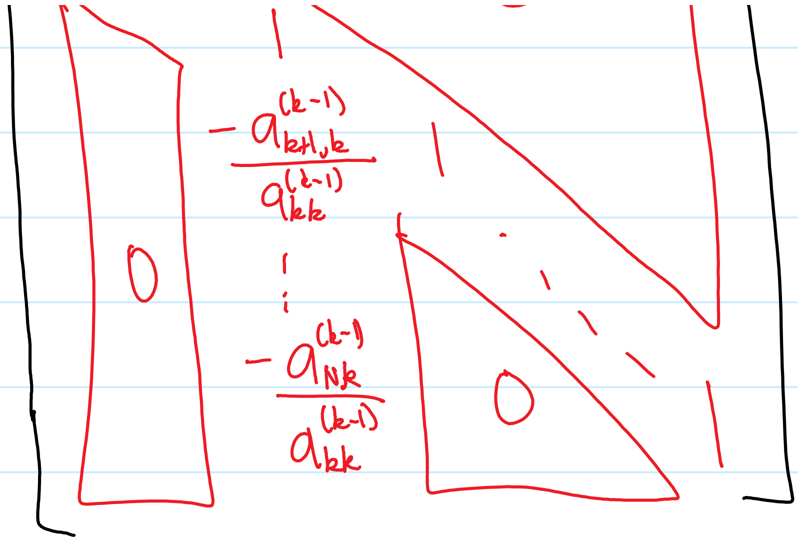
$$M^{(1)} = \begin{bmatrix} 1 & & & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ -\frac{a_{N1}}{a_{11}} & 0 & & 1 \end{bmatrix}$$

In general, at the k-th step...

$$M^{(k)} A^{(k-1)} = A^{(k)}$$

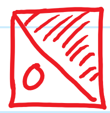
where

$$M^{(k)} = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & (k-1) & \ddots \end{bmatrix}$$



The effect of left-multiplying $A^{(k-1)}$ by $M^{(k)}$ is to eliminate x_k from rows $k+1$ to N.

At the final step: $A^{(N-1)}$ is upper-Δ



Recall that $M^{(k)}A^{(k-1)} = A^{(k)}$

For $k=1 \Rightarrow$ $M^{(1)}A = A^{(1)}$

$$M^{(2)}\left(M^{(1)}A\right) = M^{(2)}A^{(1)} = A^{(2)}$$

$$\vdots$$

$$M^{(N-1)} \cdots M^{(1)} A = A^{(N-1)} \equiv U$$

$$\Rightarrow A = \left[M^{(N-1)} \cdots M^{(1)}\right]^{-1} U$$

<u>Amazing Facts</u>

1) If B and C are lower-Δ and unit diagonal, then so is BC.
2) If B is lower-Δ and unit diagonal, then so is B^{-1}.

By fact (1), $M^{(N-1)} \cdots M^{(1)}$ is lower-Δ unit-diag.

By fact (2), $\left[M^{(N-1)} \cdots M^{(1)}\right]^{-1}$ is lower-Δ unit-diag.

By fact (2), $[M^{(N-1)} \cdots M^{(1)}]^{-1}$ is lower-Δ unit-diag.

Define $L \equiv [M^{(N-1)} \cdots M^{(1)}]^{-1}$

<u>Theorem</u>: $A = LU$

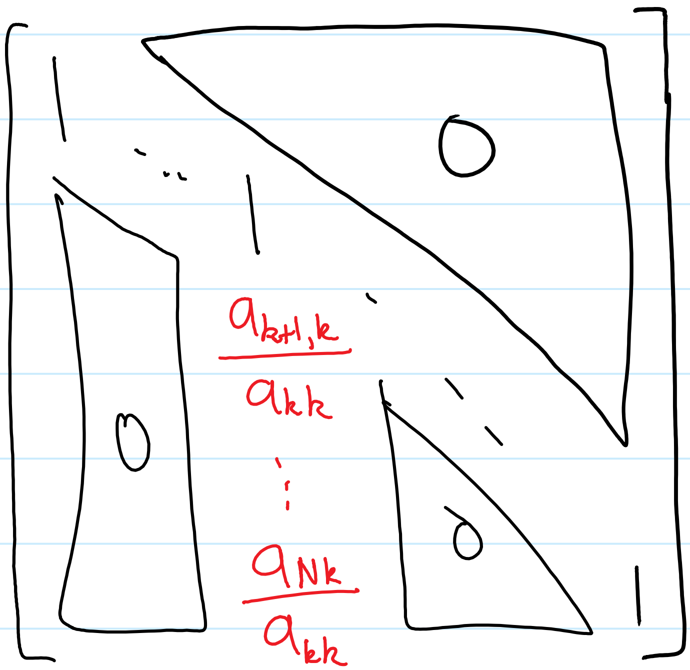
L is lower-Δ & unit diagonal

U is upper-Δ.

<u>Properties of $M^{(k)}$:</u>

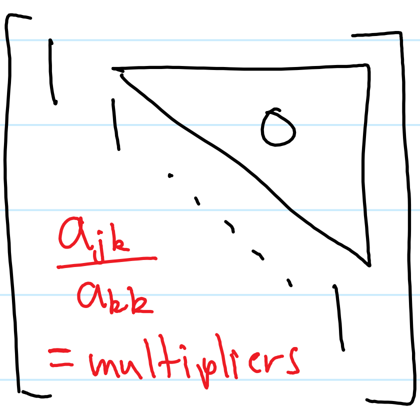
1)

$(M^{(k)})^{-1} =$



2) $L = [M^{(N-1)} \cdots M^{(1)}]^{-1} = (M^{(1)})^{-1} \cdots (M^{(N-1)})^{-1}$

$=$



i.e. the k^{th} col. of L is the k^{th} col. of $(M^{(k)})^{-1}$.

eg. $\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$

Note that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}$ is not so straight forward.

More precisely, write L_{jk} as

$$L_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j<k \\ \frac{a_{jk}}{a_{kk}} & \text{if } j>k \end{cases} \qquad \begin{matrix} k=1,\dots,N \\ j=1,\dots,N \end{matrix}$$

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 6 & -5 \\ 6 & 13 & 16 \end{bmatrix} \xrightarrow[\textcircled{3}-\frac{6}{2}\textcircled{1}]{\textcircled{2}-\frac{-4}{2}\textcircled{1}} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 16 & 7 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \qquad \xrightarrow{\textcircled{3}-\frac{16}{4}\textcircled{2}} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix} = U$$

Check:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 6 & -5 \\ 6 & 13 & 16 \end{bmatrix}$$

Goal: To stabilize LU factorization using row pivoting.

Stability of LU Factorization

In LU factorization, a problem arises when we have:

(1) a zero pivot i.e. $a_{kk}^{(k-1)} = 0$

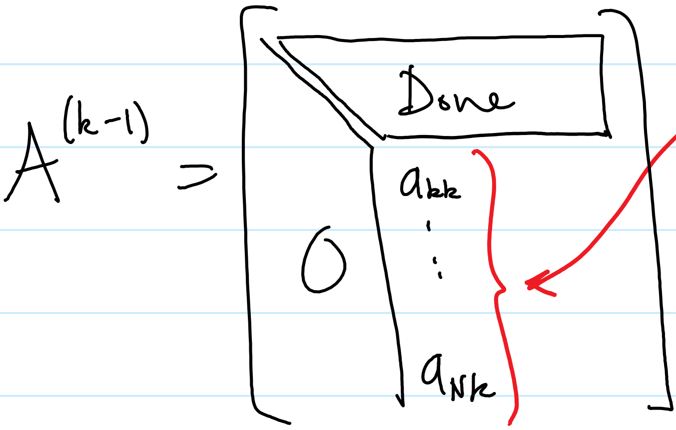
$\Rightarrow$ multipliers $\frac{a_{jk}}{a_{kk}}$ are undefined

(2) $a_{kk}^{(k-1)} \approx 0$

$\Rightarrow$ multipliers become large

$\Rightarrow$ calculations become unstable

Pivoting



find largest in magnitude.

Find $\max_{k \le j \le N} \left| a_{jk}^{(k-1)} \right| = \left| a_{k^*k}^{(k-1)} \right|$

Swap rows k^* and k, and continue.

Note: $a_{k^*k}^{(k-1)} \neq 0$

Otherwise, $a_{jk}^{(k-1)} = 0 \quad \forall\, k \le j \le N$

$\Rightarrow A$ is singular

As mentioned earlier, these row-swapping operations can be represented by matrix multiplication by a permutation matrix, P.
eg. to swap rows i & j, simply swap rows i & j in the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{② \& ③}]{\text{Swap}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P$$

Let's put it all together.

eg. $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 2 & 2 & 8 \end{bmatrix}$

$$\Big\downarrow \text{swap ① \& ②}$$

$$\begin{bmatrix} 4 & 16 & 64 \\ 1 & 1 & 1 \\ 2 & 2 & 8 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow$$

$$\begin{array}{c} \\ ② - \frac{1}{4}① \\ ③ - \frac{2}{4}① \end{array} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -3 & -15 \\ 0 & -6 & -24 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & & 1 \end{bmatrix}$$

$$\Big\downarrow \text{swap ② \& ③}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow \text{② \& ③}$$

$$\begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & -3 & -15 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\Big\downarrow$$

$$③ - \frac{-3}{-6}② \quad \begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 16 & 64 \\ 0 & -6 & -24 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 2 & 2 & 8 \end{bmatrix}$$

$$L \qquad U \qquad = \qquad P \qquad A$$

Example:

$$A = \begin{bmatrix} -6 & 27 & -42 & -15 \\ -24 & 12 & 24 & -12 \\ -12 & 14 & -10 & 20 \\ -6 & -9 & 42 & 15 \end{bmatrix}$$

$$\Big\downarrow \text{ swap ① \& ②}$$

$$\begin{bmatrix} -24 & 12 & 24 & -12 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 12 & 24 & -12 \\ -6 & 27 & -42 & -15 \\ -12 & 14 & -10 & 20 \\ -6 & -9 & 42 & 15 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{c} \\ ② - \frac{1}{4} ① \\ ③ - \frac{1}{2} ① \\ ④ - \frac{1}{4} ① \end{array} \begin{bmatrix} -24 & 12 & 24 & -12 \\ \frac{1}{4} & 24 & -48 & -12 \\ \frac{1}{2} & 8 & -22 & 26 \\ \frac{1}{4} & -12 & 36 & 18 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{c} \\ \\ ③ - \frac{1}{3} ② \\ \\ \end{array} \begin{bmatrix} -24 & 12 & 24 & -12 \\ \frac{1}{4} & 24 & -48 & -12 \\ \frac{1}{2} & \frac{1}{3} & -6 & 30 \\ \frac{1}{4} & -\frac{1}{2} & 12 & 12 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow \text{ swap } ③ \text{ \& } ④$$

$$\begin{bmatrix} -24 & 12 & 24 & -12 \\ \frac{1}{4} & 24 & -48 & -12 \\ \frac{1}{4} & -\frac{1}{2} & 12 & 12 \\ \frac{1}{2} & \frac{1}{3} & -6 & 30 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -24 & 12 & 24 & -12 \\ \frac{1}{4} & 24 & -48 & -12 \\ \frac{1}{4} & -\frac{1}{2} & 12 & 12 \\ \frac{1}{2} & \frac{1}{3} & -\frac{1}{2} & 36 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -6 & 27 & -42 & -15 \\ -24 & 12 & 24 & -12 \\ -12 & 14 & -10 & 20 \\ -6 & -9 & 42 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} -24 & 12 & 24 & -12 \\ 0 & 24 & -48 & -12 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & 0 & 36 \end{bmatrix}$$