Semi-Algebraic Systems, Complexity and Computational Lax Conjectures

Rafael Oliveira
University of Waterloo

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  But it could also not be! :)

Overview

- Introduction
  - Hyperbolic Polynomials
  - Hyperbolicity Cones
  - Semidefinite Programming & Spectrahedral Representations
  - Motivation
  - Previous Work
- Our Results
  - Main Result: Conditional Lower Bounds for Spectrahedral Representations
  - General Lax Conjecture: Equivalent Formulation
- Conclusion & Open Problems

Hyperbolic Polynomials

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Definition (Hyperbolic Polynomials)

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ is hyperbolic with respect to a point $\mathbf{e} \in \mathbb{R}^m$ if

- $h(\mathbf{e}) > 0$,
- for every vector $\mathbf{a} \in \mathbb{R}^m$, the univariate polynomial $f(t) := h(t\mathbf{e} - \mathbf{a})$ only has real zeros.

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Example

- $h(\mathbf{x}) = x_1 \cdot x_2 \cdots x_n$, $\mathbf{e} = (1, \ldots, 1)$
- $m = \binom{n+1}{2}$, X symmetric $n \times n$ matrix, $\mathbf{e} = I_n$

$$h(X) = det(X)$$

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$$\Lambda_+(h, \mathbf{e}) = \{\mathbf{a} \in \mathbb{R}^m \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative}\}$$

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Theorem ([Gårding, 1959])

- *$\Lambda_+(h, \mathbf{e})$ is a closed convex cone*
- *Equivalent definition of $\Lambda_+(h, \mathbf{e})$: closure of connected component of $\{\mathbf{a} \in \mathbb{R}^m \mid h(\mathbf{a}) \neq 0\}$ that contains $\mathbf{e}$.*

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- Origins in PDE in works of Petrovsky and Gårding.
- Convex structure can be used for optimization **[Güler, 1997]**!
- Recent applications in combinatorics and optimization **[Gurvits, 2004, Gurvits Leake 2021]**.

Hyperbolic Programming

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Given $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^m$, a hyperbolic program is the following minimization problem:

$$\begin{aligned} \inf \ & \mathbf{c}^\dagger \mathbf{x} \\ \text{s.t. } & \mathbf{x} \in \Lambda_+(h, \mathbf{e}) \end{aligned}$$

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Remark

Hyperbolic programming generalizes Linear Programming (LP) and Semidefinite Programming (SDP)!

- $h(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_m(\mathbf{x})$ (LPs)
- $h(\mathbf{x}) = \det(\sum A_i x_i)$, with A_i symmetric (SDPs)

Spectrahedral Sets & SDPs[1]

Definition (Spectrahedral Sets)

A convex set $S \subseteq \mathbb{R}^m$ is spectrahedral if it can be defined by linear matrix inequalities (LMIs). That is, there exists $d \in \mathbb{N}$ and $d \times d$ symmetric matrices $A_1, \ldots, A_m, B$ such that

$$S = \{\mathbf{c} \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq B\}.$$

S has non-empty interior if there is $\mathbf{e} \in S$ such that $\sum_i e_i \cdot A_i \succ B$.

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Open Question (General Lax Conjecture)

Is every hyperbolicity cone a spectrahedral set?

Relates the qualitative generality of HPs compared with SDPs.

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General Lax Conjecture

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Open Question (Explicit "hard" hyperbolicity cone)

Is there explicit (poly degree) hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?

Previous Work

Theorem (Non-Explicit Lower Bounds **[RRSW, 2019]**)

Exponential lower bounds on the dimension of minimal spectrahedral representations of non-explicit hyperbolicity cones (which are known to be spectrahedral).

- *Exponential lower bounds for some polynomial in a large set of hyperbolic polynomials*
- *Carefully chosen perturbations of elementary symmetric polynomial*

Previous Work

Theorem (Explicit Linear Lower Bounds **[Kummer, 2016]**)

Optimal lower bounds on the dimension of minimal spectrahedral representations of explicit hyperbolicity cones of quadratic polynomials.

- *Linear lower bounds (on number of variables) for Lorentz cone*

$$h(\mathbf{x}) = x_0^2 - x_1^2 - \cdots - x_n^2$$

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No superpoly lower bound for explicit polynomials.

Main Result: Conditional Lower Bounds

Definition (Matching Polynomial **[Amini 2019]**)

Let $G(V,E)$ be an undirected graph $\mathbf{x} = (x_v)_{v\in V},\ \mathbf{w} = (w_e)_{e\in E}$ be indeterminates.

- $\mathcal{M}(G)$ be the set of all matchings of G, $\mathcal{M}(G) \subseteq 2^E$
- for $M \in \mathcal{M}(G)$ let $V(M)$ be the vertices in this matching

$$\mu_G(\mathbf{x},\mathbf{w}) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} \cdot \prod_{v\notin V(M)} x_v \cdot \prod_{e\in M} w_e^2.$$

Amini showed that this polynomial is hyperbolic and the hyperbolicity cone of μ_G is spectrahedral.

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Theorem (Lower Bound **[O. 2020]**)

If $G = K_{n,n}$ is the complete bipartite graph, then the minimal spectrahedral representation of the hyperbolicity cone of μ_G is superpolynomial, assuming that VP $\neq$ VNP.

General Lax Conjecture - Equivalent Formulation

$h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^m$, does there exist $d \in \mathbb{N}$ and symmetric $d \times d$ matrices $A_1, \ldots, A_m$ such that

$$\Lambda_+(h, \mathbf{e}) = \{\mathbf{c} \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq 0\}$$

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A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ has a definite determinantal representation at $\mathbf{e} \in \mathbb{R}^m$ if there are symmetric matrices $A_1, \ldots, A_m$ s.t.:

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Proposition (General Lax Conjecture - Equivalent Formulation)

For each $h(\mathbf{x})$ hyperbolic at $\mathbf{e}$, there is $q(\mathbf{x})$ hyperbolic at $\mathbf{e}$, s.t.:

1. $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$
2. $h(\mathbf{x}) \cdot q(\mathbf{x})$ *has a definite determinantal representation.*

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  Any other polynomial defining variety must be a multiple of it
- Equivalent formulation of Lax conjecture + Kaltofen yield lower bound.

Conclusion & Open Questions

This work: first superpoly lower bound on the size of any spectrahedral representation for **explicit** polynomial (p-degree) (assuming VP $\neq$ VNP).

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Open Question (Quantitative Approximate General Lax Conjecture)

Is there an explicit hyperbolicity cone for which any approximate spectrahedral representation of it requires matrices of super polynomial dimension?

Open Question (General Lax Conjecture)

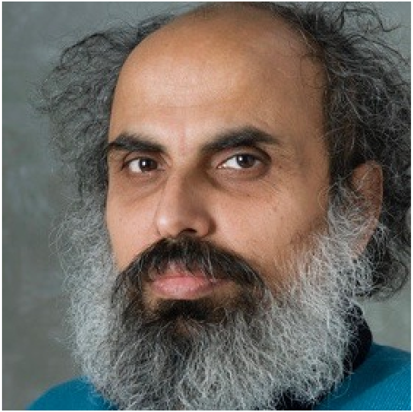
Are all hyperbolicity cones spectrahedral?

Open Question (Extended Formulations?)

Is there an explicit hyperbolicity cone for which any spectrahedral shadow representation of it requires matrices of super polynomial dimension?

Last question is open even for non-explicit polynomials.

And many more... this is just the beginning of the rabbit hole.



Bold conjecture time!

Targeted Conjectures

1. $\mathcal{S}(F) :=$ algebraic formula size for F
2. $\mathcal{S}_{hom}(F) :=$ homogeneous formula size F
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Conjecture

$$\mathcal{S}_{\Lambda}(h) = \text{poly}(\mathcal{S}_{hom}(h), \mathcal{S}_{mon}(h))$$

and

$$\mathcal{S}_{\pi,\Lambda}(h) = \text{poly}(\mathcal{S}(h))$$

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