Computational Lax Conjectures

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Overview

- Introduction
  - Hyperbolic Polynomials
  - Hyperbolicity Cones
  - Semidefinite Programming & Spectrahedral Representations
  - Previous Work

- Our Results
  - Ramanujan Detour - Matching Polynomial
  - General Lax Conjecture: Equivalent Formulation
  - Main Result: Conditional Lower Bounds for Spectrahedral Representations

- Conclusion & Open Problems

Hyperbolic Polynomials

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A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ is hyperbolic with respect to a point $\mathbf{e} \in \mathbb{R}^m$ if

- $h(\mathbf{e}) > 0$,
- for every vector $\mathbf{a} \in \mathbb{R}^m$, the univariate polynomial $f(t) := h(t\mathbf{e} - \mathbf{a})$ only has real zeros.

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Example

- $h(\mathbf{x}) = x_1 \cdot x_2 \cdots x_n$, $\mathbf{e} = (1, \ldots, 1)$
- $m = \binom{n+1}{2}$, X symmetric $n \times n$ matrix, $\mathbf{e} = I_n$

$$h(X) = det(X)$$

Hyperbolicity Cones

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Given $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^m$, its hyperbolicity cone is

$$\Lambda_+(h, \mathbf{e}) = \{\mathbf{a} \in \mathbb{R}^m \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative}\}$$

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Theorem (**[Gårding, 1959]**)

- $\Lambda_+(h, \mathbf{e})$ *is a closed convex cone*
- *Equivalent definition of* $\Lambda_+(h, \mathbf{e})$*: closure of connected component of* $\{\mathbf{a} \in \mathbb{R}^m \mid h(\mathbf{a}) \neq 0\}$ *that contains* $\mathbf{e}$*.*

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- Origins in PDE in works of Petrovsky and Gårding.
- Convex structure can be used for optimization **[Güler, 1997]**!
- Recent applications in combinatorics and optimization **[Gurvits, 2004, Gurvits Leake 2021]**.

Hyperbolic Programming

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Given $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^m$, a hyperbolic program is the following minimization problem:

$$\begin{aligned} \inf \ & \mathbf{c}^\dagger \mathbf{x} \\ \text{s.t. } & \mathbf{x} \in \Lambda_+(h, \mathbf{e}) \end{aligned}$$

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Remark

Hyperbolic programming generalizes Linear Programming (LP) and Semidefinite Programming (SDP)!

- $h(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_m(\mathbf{x})$ (LPs)
- $h(\mathbf{x}) = \det(\sum A_i x_i)$, with A_i symmetric (SDPs)

Spectrahedral Sets & SDPs[1]

Definition (Spectrahedral Sets)

A convex set $S \subseteq \mathbb{R}^m$ is spectrahedral if it can be defined by linear matrix inequalities (LMIs). That is, there exists $d \in \mathbb{N}$ and $d \times d$ symmetric matrices $A_1, \ldots, A_m, B$ such that

$$S = \{\mathbf{c} \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq B\}.$$

S has non-empty interior if there is $\mathbf{e} \in S$ such that $\sum_i e_i \cdot A_i \succ B$.

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Open Question (General Lax Conjecture)

Is every hyperbolicity cone a spectrahedral set?

Relates the qualitative generality of HPs compared with SDPs.

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General Lax Conjecture

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Can we get quantitative aspects between them?

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Is there a (poly degree) hyperbolicity cone which is "simple", but any spectrahedral representation of it requires matrices of large dimension?

Open Question (Explicit "hard" hyperbolicity cone)

Is there explicit (poly degree) hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?

Previous Work

Theorem (Non-Explicit Lower Bounds **[RRSW, 2019]**)

Exponential lower bounds on the dimension of minimal spectrahedral representations of non-explicit hyperbolicity cones (which are known to be spectrahedral).

- *Exponential lower bounds for some polynomial in a large set of hyperbolic polynomials*
- *Carefully chosen perturbations of elementary symmetric polynomial*

Previous Work

Theorem (Explicit Linear Lower Bounds **[Kummer, 2016]**)

Optimal lower bounds on the dimension of minimal spectrahedral representations of explicit hyperbolicity cones of quadratic polynomials.

- *Linear lower bounds (on number of variables) for Lorentz cone*

$$h(\mathbf{x}) = x_0^2 - x_1^2 - \cdots - x_n^2$$

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No superpoly lower bound for explicit polynomials.

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Hyperbolicity of Matching Polynomial

Definition (Matching Polynomial **[Amini 2019]**)

Let $G(V, E)$ be an undirected graph $\mathbf{x} = (x_v)_{v \in V}$, $\mathbf{w} = (w_e)_{e \in E}$ be indeterminates.

- $\mathcal{M}(G)$ be the set of all matchings of G, $\mathcal{M}(G) \subseteq 2^E$
- for $M \in \mathcal{M}(G)$ let $V(M)$ be the vertices in this matching

$$\mu_G(\mathbf{x}, \mathbf{w}) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \cdot \prod_{v \notin V(M)} x_v \cdot \prod_{e \in M} w_e^2.$$

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Amini: μ_G is hyperbolic and the hyperbolicity cone of μ_G is spectrahedral.

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Elementary proof of hyperbolicity using (multi-branched) continued fractions!

Ask me to show you after the talk :)

General Lax Conjecture - Equivalent Formulation

$h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^m$, does there exist $d \in \mathbb{N}$ and symmetric $d \times d$ matrices $A_1, \ldots, A_m$ such that

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Definition (Definite Determinantal Representations)

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ has a definite determinantal representation at $\mathbf{e} \in \mathbb{R}^m$ if there are symmetric matrices $A_1, \ldots, A_m$ s.t.:

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Proposition (General Lax Conjecture - Equivalent Formulation)

For each $h(\mathbf{x})$ hyperbolic at $\mathbf{e}$, there is $q(\mathbf{x})$ hyperbolic at $\mathbf{e}$, s.t.:

1. $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$
2. $h(\mathbf{x}) \cdot q(\mathbf{x})$ *has a definite determinantal representation.*

Main Result: Conditional Lower Bounds

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Theorem (Lower Bound **[O. 2020]**)

If $G = K_{n,n}$ is the complete bipartite graph, then the minimal spectrahedral representation of the hyperbolicity cone of μ_G is superpolynomial, assuming that VP $\neq$ VNP.

Factoring and Circuit Size

Theorem (Factors are closed in VP **[Kaltofen 1989]**)

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- Equivalent formulation of Lax conjecture + Kaltofen yield lower bound.

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Is there an explicit (poly degree) hyperbolicity cone for which any spectrahedral shadow representation of it requires matrices of super polynomial dimension?

Last question is open even for non-explicit polynomials.

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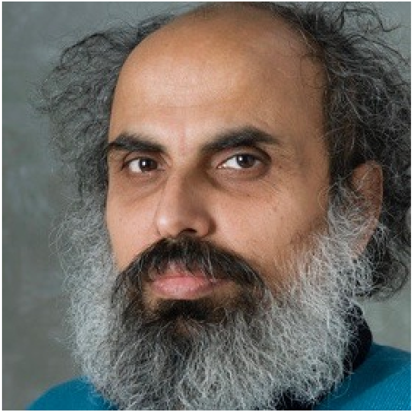
Open Question (Continued Fractions - Open Ended)

Could the multi-branched partial fraction method prove the Lax Conjecture?



Bold conjecture time!

Targeted Conjectures

1. $\mathcal{S}(F) :=$ algebraic formula size for F
2. $\mathcal{S}_{hom}(F) :=$ homogeneous formula size F
3. if $F \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$, define $\mathcal{S}_{mon}(F)$ as the minimum size of a monotone formula computing F

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- $\mathcal{S}_{\Lambda}(h) :=$ spectrahedral complexity of $\Lambda(h, \mathbf{e})$
- $\mathcal{S}_{\pi,\Lambda}(h) :=$ spectrahedral shadow complexity of $\Lambda(h, \mathbf{e})$

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- $\mathcal{S}_{\Lambda}(h)$:= spectrahedral complexity of $\Lambda(h, \mathbf{e})$
- $\mathcal{S}_{\pi,\Lambda}(h)$:= spectrahedral shadow complexity of $\Lambda(h, \mathbf{e})$

Conjecture

$$\mathcal{S}_{\Lambda}(h) = \text{poly}(\mathcal{S}_{hom}(h), \mathcal{S}_{mon}(h))$$

and

$$\mathcal{S}_{\pi,\Lambda}(h) = \text{poly}(\mathcal{S}(h))$$

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