Conditional Lower Bounds on the Spectrahedral Representation of Explicit Hyperbolicity Cones

Rafael Oliveira
rafael@uwaterloo.ca
University of Waterloo
Waterloo, Ontario

ABSTRACT

Over the past decade there has been growing interest on characterizing which convex cones over $\mathbb{R}^n$ are spectrahedral, that is, are a linear section of the cone of positive semidefinite matrices. This interest is largely motivated by applications in control theory, optimization and combinatorics. One particular class of convex cones of interest is the class of hyperbolicity cones, where the (still open) Generalized Lax Conjecture states that every hyperbolicity cone is spectrahedral. Recent works [1, 2] have established that the hyperbolicity cones of the elementary symmetric polynomials and the homogeneous multivariate matching polynomial are spectrahedral, but the question of whether there exists an efficient spectrahedral representation for such cones remains open. Previous work [11] has provided exponential lower bounds on the spectrahedral representation of *non-explicit* hyperbolicity cones which are known to be spectrahedral. The current best lower unconditional bounds for *explicit cones* are the *linear* lower bounds proved by [7].

In this paper we establish the first *superpolynomial* hardness of the minimal spectrahedral representation for an *explicit family of hyperbolicity cones*, assuming Valiant's *VP vs VNP* conjecture is true, that is, that the permanent polynomial cannot be computed by algebraic circuits of polynomial size. More precisely, we prove that the hyperbolicity cone of Amini's *homogeneous matching polynomial* must require superpolynomial spectrahedral representations, assuming that Valiant's conjecture is true. This is the first work providing a (conditional) superpolynomial lower bound on the spectrahedral representation of an explicit hyperbolicity cone.

CCS CONCEPTS

• **Theory of computation** → **Algebraic complexity theory**; **Convex optimization**; *Semidefinite programming*; • **Mathematics of computing** → *Convex optimization*.

KEYWORDS

Algebraic Complexity, Hyperbolic Polynomials, Hyperbolicity Cones, Convex Optimization, Semidefinite Programming

ISSAC '20, July 20–23, 2020, Kalamata, Greece

ACM ISBN 978-1-4503-7100-1/20/07…$15.00
https://doi.org/10.1145/3373207.3404010

ACM Reference Format:
Rafael Oliveira. 2020. Conditional Lower Bounds on the Spectrahedral Representation of Explicit Hyperbolicity Cones. In *International Symposium on Symbolic and Algebraic Computation (ISSAC '20), July 20–23, 2020, Kalamata, Greece.* ACM, New York, NY, USA, 6 pages. https://doi.org/10.1145/3373207.3404010

1 INTRODUCTION

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector of variables $x_1, \ldots, x_n$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be a vector of elements $a_1, \ldots, a_n$ from $\mathbb{R}$. A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to a direction $\mathbf{e} := (e_1, \ldots, e_n) \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$ and for all vectors $\mathbf{a} \in \mathbb{R}^n$, the univariate polynomial $f(t) := h(t\mathbf{e} - \mathbf{a})$ only has real zeros. By a result due to Gårding [3], each hyperbolic polynomial $h(\mathbf{x})$ defines a *hyperbolicity cone*, a closed convex cone denoted by $\Lambda_+(h, \mathbf{e})$ and defined as

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{a} \in \mathbb{R}^n \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative}\}.$$

Gårding also showed [3] that $\Lambda_+(h, \mathbf{e})$ can be equivalently defined as the closure of the connected component of $\{\mathbf{a} \in \mathbb{R}^n \mid h(\mathbf{a}) \neq 0\}$ that contains $\mathbf{e}$.

Hyperbolic polynomials and hyperbolicity cones originated in the theory of PDE in the works of Petrovsky and Gårding, and are of importance in combinatorics and optimization. Hyperbolicity cones are important objects in optimization, as they generalize semidefinite cones and Güler [4] showed that one could generalize interior point methods of optimization to hyperbolicity cones. Since then the theory of hyperbolic programming has been vastly expanded, see [12] and references therein.

Despite much progress on the optimization side of hyperbolic programming, the geometric and complexity theoretic aspects of hyperbolicity cones are much less understood.

On the geometric side, an important open question is concerned with how general the class of hyperbolicity cones is. *Spectrahedral cones*, that is, linear sections of the cone of positive semidefinite matrices, form the most well-known examples of hyperbolicity cones. The generalized Lax conjecture states that every hyperbolicity cone is also a spectrahedral cone, whereas the projected Lax conjecture states that every hyperbolicity cone is a linear projection of a spectrahedral cone. Despite much recent work and some impressive progress on these conjectures [8, 10], they remain open.

The origins of these conjectures came from partial differential equations. When the number of variables of a hyperbolic polynomial is 3, say $h(x, y, z)$ is hyperbolic in direction (a, b, c), Lax conjectured [9] that any such hyperbolic polynomial could be written as a determinant of a linear combination of symmetric matrices of the form $xA + yB + zC$ such that $a \cdot A + b \cdot B + c \cdot C$ is positive definite. This conjecture certainly implies that for 3 variables, every hyperbolicity cone is a spectrahedral cone. A positive answer to this conjecture was given by Helton and Vinnikov [5].

On the complexity theoretic side, very little is known about the complexity of representing hyperbolicity cones which are known to be spectrahedral. In the recent work [11], the authors prove exponential lower bounds even for approximate spectrahedral representations of *non-explicit* hyperbolicity cones which are spectrahedral. However, prior to the present work, no superpolynomial lower bound on the spectrahedral representation for an *explicit* hyperbolicity cone which is also spectrahedral was known. In the next section we present our main result and the overview of its proof, which is given formally in the next sections.

1.1 Main result and proof overview

In this paper, we prove a conditional lower bound on the minimal spectrahedral representation of the hyperbolicity cone of an *explicit family* of spectrahedral polynomials. More precisely, we prove the following theorem:

Theorem 1.1. *There exists an explicit family of hyperbolic polynomials $\{h_n(\mathbf{x})\}_{n\geq 1}$ and directions $\{\mathbf{e}_n\}_{n\geq 1}$, where $h_n(\mathbf{x})$ has $\mathrm{poly}(n)$ variables and $\mathrm{poly}(n)$ degree, whose hyperbolicity cone $\Lambda_+(h_n, \mathbf{e}_n)$ is spectrahedral and such that any spectrahedral representation of $\Lambda_+(h_n, \mathbf{e}_n)$ must have superpolynomial size in n, assuming that VP ≠ VNP.*

High-level ideas of the proof: The high level idea guiding the proof of Theorem 1.1 comes from the combination of the four facts below:

1. Every spectrahedral cone has a corresponding definite determinantal representation. This follows by the definition of the spectrahedral cone.
2. Irreducible hyperbolic polynomials are the minimal defining polynomials of their hyperbolicity cones. This fact follows from standard results in real algebraic geometry, and a proof is given in [5, Lemma 2.1].
3. A necessary condition for the hyperbolicity cone of an irreducible hyperbolic polynomial $h(\mathbf{x})$ to be spectrahedral is the existence of a definite determinantal polynomial which is a multiple of $h(\mathbf{x})$. In Proposition 2.4 a necessary and sufficient condition is given.
4. Factors of polynomials of small degree computed by small algebraic circuits also have small algebraic circuits, as was proved in the seminal work [6].

The facts above yield a useful necessary condition for a hyperbolicity cone to have a polynomial sized spectrahedral representation, and this necessary condition comes from algebraic complexity: the hyperbolic polynomial must be computed by polynomial sized circuits! This can be seen as follows: given a hyperbolicity cone, take its minimal defining polynomial $h(\mathbf{x})$. By [5, Lemma 2.1], any other polynomial $q(\mathbf{x})$ defining the same hyperbolicity cone must be a multiple of $h(\mathbf{x})$. If the hyperbolicity cone of $h(\mathbf{x})$ is spectrahedral, then there is a definite determinantal polynomial $D(\mathbf{x})$ defining the hyperbolicity cone of $h(\mathbf{x})$. If $D(\mathbf{x})$ can be defined by polynomial sized matrices, then the polynomial $D(\mathbf{x})$ can be computed by polynomial sized circuits. Thus, Kaltofen's seminal result (item 4) tells us that $h(\mathbf{x})$ can also be computed by polynomial sized circuits!

With the necessary condition above, the proof strategy is straightforward: simply construct an explicit irreducible hyperbolic polynomial $h(\mathbf{x})$ that requires superpolynomial sized algebraic circuits to compute it, and whose hyperbolicity cone is spectrahedral. Irreducibility of $h(\mathbf{x})$ implies that it is the minimal defining polynomial of its hyperbolicity cone, by item 2 above. Hardness of $h(\mathbf{x})$ and the necessary condition given by the previous paragraph, implies that any definite determinantal representation of the hyperbolicity cone of $h(\mathbf{x})$ must have superpolynomial size.

The only task left is to construct an irreducible hyperbolic polynomial which has a spectrahedral hyperbolicity cone and that is hard to compute by algebraic circuits. And it just so happens that Amini's homogeneous matching polynomial over the complete bipartite graph has all the properties above. Amini [1] shows that the homogeneous matching polynomial has a spectrahedral hyperbolicity cone. In Section 4 we show that this polynomial is irreducible for the complete bipartite graph.

Since we do not currently know superpolynomial lower bounds on the circuit complexity of any explicit polynomial, we will prove a conditional lower bound, which is based on Valiant's conjecture that VP ≠ VNP. Valiant's conjecture can be stated as: the Permanent polynomial cannot be computed by polynomial sized circuits. Thus, to prove that Amini's homogeneous matching polynomial is hard, we prove a reduction result: we show that if the matching polynomial of the complete bipartite graph can be computed by polynomial sized circuits, then there is a polynomial sized circuit computing the Permanent.

1.2 Related Work

Much work in the past decade has focused on proving generalizations of the Lax conjecture, whose aim is to relate hyperbolicity cones to spectrahedral cones. The *generalized Lax conjecture* states that every hyperbolicity cone is spectrahedral, while the *projected Lax conjecture* states that every hyperbolicity cone is the projection of a spectrahedral cone.

In [8], the author makes progress towards the generalized Lax conjecture, proving that every smooth hyperbolic polynomial is a factor of a definite determinantal polynomial, thus establishing one part of the equivalence from Proposition 2.4. In [10], the authors prove that smooth hyperbolicity cones are projections of spectrahedra, thus showing that the projected Lax conjecture is holds for almost all hyperbolicity cones. However, in these papers the computational complexity of their constructions is still left unexplored, and the current work is a step forward in understanding the computational complexity of these hyperbolicity cones.

On the lower bounds/impossibility side, [13] proves that many compact convex semialgebraic sets in euclidean space are not projections of spectrahedra. In [11], the authors prove exponential lower bounds on the spectrahedral representations of non-explicit spectrahedral hyperbolicity cones. Their lower bounds are unconditional, albeit being non-explicit.

1.3 Organization

In Section 2 we formally define hyperbolic polynomials and their hyperbolicity cones, spectrahedral cones and definite determinantal representations, establishing the basic facts about them, as well as the interconnections between these concepts. In Section 3 we establish the basic definitions and facts that we will need from Algebraic Complexity Theory, including the irreducibility and hardness of a variant of the Permanent polynomial. In Section 4 we prove the main result of the paper, which is the conditional lower bound on the spectrahedral representation of the hyperbolicity cone of the matching polynomial. In Section 5 we conclude and present some open problems.

2 HYPERBOLIC POLYNOMIALS AND SPECTRAHEDRALITY

In this section we formally define hyperbolic polynomials, definite determinantal representations, spectrahedral representations, and establish the known relationships between these three concepts.

2.1 Hyperbolic Polynomials and Definite Determinantal Representation

In this section we formally give the main definitions and background needed from hyperbolic polynomials and definite determinantal representations which will be used in the later sections.

Definition 2.1 (Hyperbolic Polynomial)**.** A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ of degree d is hyperbolic with respect to direction $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$ and for every $\mathbf{a} \in \mathbb{R}^n$, the univariate polynomial $h(t \cdot \mathbf{e} - \mathbf{a})$ is real rooted (counting their multiplicities). That is, $h(t \cdot \mathbf{e} - \mathbf{a})$ has exactly d real roots.

Definition 2.2 (Hyperbolicity Cone)**.** If $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is a hyperbolic polynomial with respect to direction $\mathbf{e}$, its *hyperbolicity cone* is the set defined by

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{a} \in \mathbb{R}^n \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative } \}.$$

Definition 2.3 (Definite Determinantal Representation)**.** We say that a homogeneous polynomial $h(x) \in \mathbb{R}[\mathbf{x}]$ has a definite determinantal representation at $\mathbf{b} \in \mathbb{R}^n$ if there are $A_1, \ldots, A_n \in \mathrm{Sym}_d(\mathbb{R})$ and $\lambda \in \mathbb{R}^*$ such that:

(1) $\sum_{i=1}^{n} b_i \cdot A_i \succ 0$

(2) $h(\mathbf{x}) = \lambda \cdot \det\left(\sum_{i=1}^{n} x_i \cdot A_i\right)$.

Proposition 2.4 (Spectrahedral Representation Equivalent Formulation [17]). *Let $h \in \mathbb{R}[\mathbf{x}]$ be hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$. The hyperbolicity cone $\Lambda_+(h, \mathbf{e})$ is spectrahedral if, and only if, there is a hyperbolic polynomial $q \in \mathbb{R}[\mathbf{x}]$ with respect to $\mathbf{e}$ such that the following two conditions are satisfied:*

(1) $q \cdot h$ has a definite determinantal representation at $\mathbf{e}$

(2) $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$.

The following follows from [5, Lemma 2.1]. It essentially states that the hyperbolicity cone $\Lambda_+(h, \mathbf{e})$ of an irreducible hyperbolic polynomial h has the polynomial h as its *minimal defining polynomial*. That is, any other polynomial g also defining $\Lambda_+(h, \mathbf{e})$ must be a multiple of h.

Proposition 2.5 (Hyperbolic Cones of Irreducible Polynomials). *If $h \in \mathbb{R}[\mathbf{x}]$ is an irreducible and hyperbolic polynomial with respect to $\mathbf{e} \in \mathbb{R}^n$, and $q \in \mathbb{R}[\mathbf{x}]$ is a hyperbolic polynomial such that $\Lambda_+(h, \mathbf{e}) = \Lambda_+(q, \mathbf{e})$, then h divides q.*

If a hyperbolicity cone $\Lambda_+(h, \mathbf{e})$ is spectrahedral, i.e. a linear section of the positive semidefinite cone, let

$$\Lambda_+(h, \mathbf{e}) = \left\{\mathbf{a} \in \mathbb{R}^n \mid \sum_{i=1} a_i \cdot A_i \succeq 0\right\}$$

be any spectrahedral representation of the hyperbolicity cone, where $A_i \in Sym_D(\mathbb{R})$ are real symmetric matrices of dimension D.

In this case, we have that $P(\mathbf{x}) = \mathrm{Det}(\sum_{i=1}^{n} A_i \cdot x_i)$ is a hyperbolic poylnomial at $\mathbf{e}$ such that $\Lambda_+(P, \mathbf{e}) = \Lambda_+(h, \mathbf{e})$. Thus, if $h(\mathbf{x})$ is an irreducible polynomial, by Proposition 2.5, we must have that $h(\mathbf{x})$ divides $P(\mathbf{x})$. We will need this fact in the proof of our main result in Section 4.

2.2 Homogeneous Multivariate Matching Polynomial

In this section we describe our candidate hard polynomial, which was first defined in [1, Definition 2.1] as a multivariate generalization of the univariate matching polynomial from algebraic combinatorics, and as a variant on the multivariate matching polynomial of Heilmann and Lieb.

Definition 2.6 (Homogeneous Multivariate Matching Polynomial [1])**.** Let $G(V, E)$ be an undirected graph, $\mathbf{x} = (x_v)_{v \in V}$ and $\mathbf{w} = (w_e)_{e \in E}$ be indeterminates. The *homogeneous multivariate matching polynomial* is defined by

$$\mu_G(\mathbf{x}, \mathbf{w}) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \cdot \prod_{v \notin V(M)} x_v \cdot \prod_{e \in M} w_e^2, \qquad (1)$$

where in the equation above $\mathcal{M}(G)$ is the set of all matchings of G (including the empty set), M is a matching of G (the collection of edges forming the matching), $V(M)$ is the set of vertices participating in the matching M and $|M|$ is the number of edges in the matching.

Remark 2.7. *Note that if a graph G has perfect matchings, they are captured by $\mu_G(\mathbf{x}, \mathbf{w})$ by setting $\mathbf{x} = \mathbf{0}$. That is,*

$$\mu_G(\mathbf{0}, \mathbf{w}) = \sum_{M \text{ is perfect matching}} (-1)^{|M|} \cdot \prod_{e \in M} w_e^2$$

Throughout this section, we let $\mathbf{e} := (\mathbf{1}_V, \mathbf{0}_E)$ be the direction given by the all one's vector in the variables $(x_v)_{v \in V}$ and the zero vector in the variables $(w_e)_{e \in E}$. In [1, Theorem 2.12], Amini shows that the hyperbolicity cone $\Lambda_+(\mu_G, \mathbf{e})$ is spectrahedral.

Proposition 2.8 (Spectrahedrality of Matching Polynomial [1]). *The hyperbolicity cone $\Lambda_+(\mu_G(\mathbf{x}, \mathbf{w}), \mathbf{e})$ is spectrahedral.*

From the fact above, together with Proposition 2.4, we obtain the following corollary.

Corollary 2.9. *There exists a hyperbolic polynomial $q \in \mathbb{R}[\mathbf{x}, \mathbf{w}]$ w.r.t. direction $\mathbf{e}$ such that the polynomial $q \cdot \mu_G(\mathbf{x}, \mathbf{w})$ has a definite determinantal representation and $\Lambda_+(\mu_G, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$.*

3 ALGEBRAIC COMPLEXITY

In this section, we define the basic notions of algebraic complexity and establish the basic facts which we will need for the proof of our main theorem in the next section. We start with the definition of an algebraic circuit, which can be found in [14].

Definition 3.1 (Algebraic Circuits)**.** An algebraic circuit Φ over a field $\mathbb{F}$ and a set of variables $\mathbf{x} = (x_1, \ldots, x_n)$ is a directed acyclic graph defined as follows. The vertices of Φ are the gates of the circuit, and each gate of indegree 0 is labeled by either a variable from $\mathbf{x}$ or by a field element from $\mathbb{F}$. Every other gate in Φ is labeled by either $+, \times$ and has indegree 2.

From the definition above, one can see that an algebraic circuit computes polynomials in a natural way. Each input gate is either a variable or a field element, and a + gate computes the polynomial given by the sum of its input gates, and a $\times$ gate computes the product of its input gates. We say that a circuit Φ computes a polynomial p if there is a gate of Φ which computes the polynomial p.

The size of an algebraic circuit is defined as the number of gates in the circuit. The formal degree of a circuit Φ is defined inductively as follows: an input gate of Φ has degree 1 if it is a variable, and 0 otherwise. For any + gate $u = v + w$ of the circuit, we make $\deg(u) = \max\{\deg(v), \deg(w)\}$ and for a $\times$ gate $u = v \times w$ we make $\deg(u) = \deg(v) + \deg(w)$. We define the degree of Φ as the maximum degree among the degrees of the gates of Φ.

We say that a circuit Φ is a homogeneous circuit if each gate of Φ computes a homogeneous polynomial. Note that in a homogeneous circuit Φ computing a (homogeneous) polynomial p of degree d only the gates of degree $\leq d$ are needed from Φ. Hence, if we are interested in the computation of p alone, we can assume that Φ has degree d as well.

Given a polynomial $p(\mathbf{x})$, denote its homogeneous component of degree r by $H_r[p(\mathbf{x})]$. The following proposition due to [15] tells us that given an algebraic circuit of polynomial size, we can efficiently compute its low degree components with algebraic circuits. A proof can be found in [14, Theorem 2.2].

Proposition 3.2 (Complexity of Computing Homogeneous Components [15]). *If $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ can be computed by an algebraic circuit $\Phi(\mathbf{x})$ of size s, then for every $r \in \mathbb{N}$, there is a homogeneous circuit $\Psi(\mathbf{x})$ of size at most $O(r^2 s)$ computing $H_0[p(\mathbf{x})], \ldots, H_r[p(\mathbf{x})]$.*

Remark 3.3. *Note that in the proposition above, there is no requirement on the degree of the circuit Φ, while the homogeneous circuit Ψ will have degree bounded by r.*

One of the main goals of algebraic complexity theory is to classify which families of polynomials $\{p_n\}_{n\geq 1}$ where $p_n \in \mathbb{F}[x_1, \ldots, x_n]$ can be computed by a family of algebraic circuits $\{\Phi_n\}_{n\geq 1}$ of polynomial size. The theory has mostly been concerned with families of polynomials $\{p_n\}_{n\geq 1}$ with $\deg(p_n)$ being a polynomial function of n.

For such families of polynomials having polynomial degree in the number of variables, the class of families of polynomials which can be computed by a family of algebraic circuits of polynomial size is denoted by VP. This is the class of "efficiently computable" polynomials.

One of the most important family of polynomials which is in VP is the family defined by the determinant polynomial: given an $n \times n$ symbolic matrix X,

$$\mathrm{Det}_n(X) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} X_{i\sigma(i)}.$$

Another important class of families of polynomials is the class denoted by VNP, which is the algebraic analogue of the class NP, and informally speaking is the class of families of polynomials which can be "defined efficiently." For a more precise definition see [14, Definition 1.3].

There is a beautiful theory of completeness and reductions for these algebraic classes, analogue to the theory developed in the boolean setting for P and NP, whose origins trace back to the seminal work of Valiant [16]. One of the major open problems in algebraic complexity theory, posed by Valiant, is whether the classes VP and VNP are different or not.

One complete family of polynomials in VNP is defined by the permanent polynomial: given an $n \times n$ symbolic matrix X,

$$\mathrm{Per}_n(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma(i)},$$

and therefore the VP versus VNP question can be stated as:

Conjecture 3.4 (Valiant's VP ≠ VNP Conjecture). *The family defined by the permanent polynomials $\{\mathrm{Per}_n(X)\}n \geq 1$ cannot be computed by circuits in VP.*

For the sake of conciseness, we shall from now on refer to a family of polynomials simply by one of its elements. For instance, when talking about the family defined by the permanent polynomials of degree n, we shall simply talk about the polynomial $\mathrm{Per}_n(X)$. The parameter defining the family of polynomials is n. Thus, we will refer to the polynomial $\mathrm{Per}_n(X)$ and the family $\{\mathrm{Per}_n(X)\}_{n\geq 1}$ interchangeably.

The class VP enjoys many closure properties under fundamental algebraic operations. One of its most remarkable was proved in the seminal work of Kaltofen [6] and states that the class VP is closed under factorization.

Proposition 3.5 (Factors are Closed in VP [6]). *If a polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ of degree d can be computed by an algebraic circuit of size s, then any factor $g(\mathbf{x})$ of the polynomial $p(\mathbf{x})$ can be computed by an algebraic circuit of size* $\mathrm{poly}(n, s, d)$.

We now proceed to establishing two lemmas that shall be important for us in the subsequent sections. From now on, we will be working over the base field $\mathbb{R}$. The first lemma establishes the VNP-hardness of a particular polynomial: the squared permanent polynomial, which is defined below.

Lemma 3.6 (Complexity of the squared Permanent). *Let $W = (w_{i,j}^2)_{i,j=1}^n$ be a symbolic matrix over the variables $\mathbf{w} = (w_{i,j})_{i,j=1}^n$. If VP ≠ VNP then any algebraic circuit computing $\mathrm{Per}_n(W)$ must have superpolynomial size.*

Proof. Assume, for the sake of contradiction, that there is a circuit $\Phi(\mathbf{w})$ of size $O(n^c)$ computing $\mathrm{Per}_n(W)$, where $c \in \mathbb{Z}$ is a positive constant.

Let $u_{i,j} = (1 - x_{i,j})^{1/2}$. Then, $\Phi(\mathbf{u}) = \mathrm{Per}_n(J - X)$, where J is the all-ones matrix and $X = (x_{i,j})$ is a pure symbolic matrix.

Each $u_{i,j}$ is a univariate real analytic function on the variable $x_{i,j}$ over the ball of radius $1/2$ around the origin. Take the Taylor expansion of $u_{i,j}$ around $x_{i,j} = 0$. Call this Taylor series $v_{i,j}$. The truncated Taylor series $v_{i,j}$, truncated at degree n, can be computed by an algebraic circuit of size $O(n)$, as it is an univariate polynomial of degree n. Let $T_{i,j}$ be the truncation of $v_{i,j}$ at degree n.

Letting $T = (T_{i,j})_{i,j=1}^n$, we have that

$$(-1)^n \cdot \mathrm{Per}_n(X) = H_n[\mathrm{Per}_n(J - X)] = H_n[\Phi(T)].$$

Note that $\Phi(T)$ is a circuit of size[1] $O(n^{c+3})$, as we replaced each variable $w_{i,j}$ in the cicuit $\Phi(\mathbf{w})$ by the truncated Taylor expansion $T_{i,j}$ of $u_{i,j}$, and we saw that each $T_{i,j}$ can be computed by a circuit of size $O(n)$. As there are n^2 such Taylor expansions, the size is $O(n^{c+3})$.

By applying Proposition 3.2, the homogeneous part of degree n of $\Phi(T)$ can be computed by a homogeneous circuit of size $O(n^2 \cdot n^{c+3}) = O(n^{c+5})$ and degree n. This implies that $\mathrm{Per}(X) \in \mathsf{VP}$, which would imply that $\mathsf{VP} = \mathsf{VNP}$. ◻

We will also need to establish that the squared permanent is an irreducible polynomial. This will be important in our proof that the homogeneous matching polynomial of the complete bipartite graph is irreducible.

Lemma 3.7 (Irreducibility of the squared Permanent). *Let $n \geq 2$ and $W = (w_{i,j}^2)_{i,j=1}^n$ be a symbolic matrix over the variables $\mathbf{w} = (w_{i,j})_{i,j=1}^n$. Then the polynomial $\mathrm{Per}(W)$ is irreducible over $\mathbb{R}[\mathbf{w}]$.*

Proof. Suppose $\mathrm{Per}(W) = p(\mathbf{w}) \cdot q(\mathbf{w})$. Assume there is some entry $(i, j) \in [n]^2$ such that $p(\mathbf{w})$ is linear w.r.t. $w_{i,j}$. In this case, $q(\mathbf{w})$ is also linear w.r.t. $w_{i,j}$ and we would have $p(\mathbf{w}) = a_p \cdot w_{i,j} + b_p$ and $q(\mathbf{w}) = a_q \cdot w_{i,j} + b_q$, where $a_p, b_p, a_q, b_q \in \mathbb{R}[\mathbf{w}]$ are nonzero polynomials which do not depend on $w_{i,j}$. In this case, we have that $a_p \cdot a_q$ computes the permanent of the (i, j)-minor of W (and thus is a sum of squares polynomial) and we have that $b_p \cdot b_q$ computes another sum of squares polynomial (due to the cofactor expansion of the Permanent). This implies that $a_p \cdot a_q > 0$ for all its non-zero values, and so is $b_p \cdot b_q > 0$.

However, as $\mathrm{Per}(W) = p(\mathbf{w}) \cdot q(\mathbf{w})$, the linear term in $w_{i,j}$ in the multiplication $p(\mathbf{w}) \cdot q(\mathbf{w})$ must vanish, thus implying $a_p \cdot b_q + a_q \cdot b_p = 0$, which implies that $a_p \cdot b_q \cdot a_q \cdot b_p < 0$ for any non-zero evaluation of these polynomials, contradicting the previous paragraph.

Thus, we are left with the case where for each $w_{i,j}$, we have that either $p(\mathbf{w}) = w_{i,j}^2 \cdot a_p + b_p$ and $q(\mathbf{w}) = b_q$, where $a_p, b_p, b_q \in \mathbb{R}[\mathbf{w}]$ do not depend on $w_{i,j}$, or the other way around (q is the purely quadratic polynomial in $w_{i,j}$ whereas p is constant in $w_{i,j}$). In this case, since no linear terms on any $w_{i,j}$ appear in the factorization $\mathrm{Per}(W) = p(\mathbf{w}) \cdot q(\mathbf{w})$, this factorization after doing a change of variables $x_{i,j} = w_{i,j}^2$ yields a polynomial factorization of the usual permanent, which is known to be irreducible for $n \geq 2$. ◻

[1]The more precise bound is $O(n^{\max(c,3)})$, since the size of a composition of circuits is simply the sum of the sizes of the circuits being used.

4 COMPLEXITY OF DEFINITE DETERMINANTAL REPRESENTATIONS

In this section we prove the main result of this paper: the conditional complexity lower bound on the spectrahedral representation of the matching polynomial for the complete bipartite graph $K_{n,n}$.

For this section, we will let $\mu(\mathbf{x}, \mathbf{w}) \triangleq \mu_{K_{n,n}}(\mathbf{x}, \mathbf{w})$ and $\mathbf{e} = (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_{E(K_{n,n})})$ be the hyperbolicity direction for $\mu(\mathbf{x}, \mathbf{w})$ from Amini's theorem.

Lemma 4.1 (Complexity of Complete Bipartite Matching Polynomial). *Assuming VP ≠ VNP, that is, that the permanent polynomial has super-polynomial circuit size, then the polynomial $\mu(\mathbf{x}, \mathbf{w})$ requires super polynomial size circuits.*

Proof. Let $W = (w_{ij}^2)_{i,j=1}^n$ be a symbolic matrix. Note that $\mu(\mathbf{0}, \mathbf{w}) = \mathrm{Per}_n(W)$. By Lemma 3.6 and our assumption that $\mathsf{VP} \neq \mathsf{VNP}$, we have that $\mu(\mathbf{0}, \mathbf{w})$ requires superpolynomial-sized circuits to compute it.

If $\Phi(\mathbf{x}, \mathbf{w})$ is any algebraic circuit computing $\mu(\mathbf{x}, \mathbf{w})$ with size s (i.e., having s gates, one of them computing the polynomial $\mu(\mathbf{x}, \mathbf{w})$), the circuit $\Phi(\mathbf{0}, \mathbf{w})$, obtained by setting the input variables $\mathbf{x}$ to $\mathbf{0}$, also has size $\leq s$ and computes the polynomial $\mu(\mathbf{0}, \mathbf{w})$. As $\Phi(\mathbf{0}, \mathbf{w})$ requires superpolynomial size, by the previous paragraph, we also have that $\Phi(\mathbf{x}, \mathbf{w})$ requires superpolynomial size. ◻

Lemma 4.2 (Irreducibility of Complete Bipartite Matching Polynomial). *The polynomial $\mu(\mathbf{x}, \mathbf{w})$ is irreducible over $\mathbb{R}[\mathbf{x}, \mathbf{w}]$.*

Proof. Suppose, for the sake of contradiction, that $\mu(\mathbf{x}, \mathbf{w})$ factors. Then, there exist polynomials $p(\mathbf{x}, \mathbf{w})$ and $q(\mathbf{x}, \mathbf{w})$ such that $\mu(\mathbf{x}, \mathbf{w}) = p(\mathbf{x}, \mathbf{w}) \cdot q(\mathbf{x}, \mathbf{w})$. Consider the polynomials above in the ring $(\mathbb{R}[\mathbf{w}])[\mathbf{x}]$. As the constant coefficient of $\mu(\mathbf{x}, \mathbf{w})$ is $\mu(\mathbf{0}, \mathbf{w}) = (-1)^n \cdot \mathrm{Per}_n(W)$, which is nonzero, we must have that $p(\mathbf{0}, \mathbf{w})$ and $q(\mathbf{0}, \mathbf{w})$ are nonzero. However, by Lemma 3.7, we have that $\mathrm{Per}_n(W)$ is irreducible, which implies w.l.o.g. that $p(\mathbf{0}, \mathbf{w}) = (-1)^n \cdot \mathrm{Per}_n(W)$ and $q(\mathbf{0}, \mathbf{w}) = 1$.

Since $\mu(\mathbf{x}, \mathbf{0}) = \prod_{1 \leq i \leq 2n} x_i$ is nonzero, we must have $p(\mathbf{x}, \mathbf{0})$ and $q(\mathbf{x}, \mathbf{0})$ are nonzero. If we look at $\mu(\mathbf{x}, \mathbf{0}) = p(\mathbf{x}, \mathbf{0}) \cdot q(\mathbf{x}, \mathbf{0})$, we have that $q(\mathbf{x}, \mathbf{0})$ must either be constant or a monomial over $\mathbf{x}$. As the previous paragraph implies $q(\mathbf{0}, \mathbf{0}) = 1$, $q(\mathbf{x}, \mathbf{0})$ cannot be a non-constant monomial over $\mathbf{x}$, as that would imply $q(\mathbf{0}, \mathbf{0}) = 0$. Hence, we have that $p(\mathbf{x}, \mathbf{0}) = \prod_{1 \leq i \leq 2n} x_i$.

If $q(\mathbf{x}, \mathbf{w})$ is a non-constant polynomial, any of its non-constant monomials must depend on both $\mathbf{x}$ and $\mathbf{w}$ variables, as $q(\mathbf{0}, \mathbf{w}) = q(\mathbf{x}, \mathbf{0}) = 1$. If $q(\mathbf{x}, \mathbf{w})$ depends on some $\mathbf{x}$ variable, say x_1 w.l.o.g., write $q(\mathbf{x}, \mathbf{w}) = q_1(\mathbf{x}, \mathbf{w})x_1 + q_0(\mathbf{x}, \mathbf{y})$, where q_0 does not depend on x_1. As $\mu(\mathbf{x}, \mathbf{w})$ is linear in x_1, we must have that q is linear in x_1 and p does not depend on x_1. However, this contradicts the fact that $p(\mathbf{x}, \mathbf{0}) = \prod_{1 \leq i \leq 2n} x_i$. Hence, we conclude that $q(\mathbf{x}, \mathbf{w})$ does not depend on any $\mathbf{x}$ variable, which implies $q(\mathbf{x}, \mathbf{w}) = q(\mathbf{0}, \mathbf{w}) = 1$, which proves that $\mu(\mathbf{x}, \mathbf{w})$ is irreducible. ◻

Putting the pieces together, we can now prove our main result: assuming that $\mathsf{VP} \neq \mathsf{VNP}$, any spectrahedral representation of the hyperbolicity cone of the complete bipartite matching polynomial has superpolynomial size.

Theorem 4.3 (Hardness of Spectrahedral Representation). *Assuming that VP ≠ VNP, the following is true: any spectrahedral representation of the spectrahedral cone* $\Lambda_+(\mu, \mathbf{e})$ *of the matching polynomial* $\mu_{K_{n,n}}(\mathbf{x}, \mathbf{w})$ *has superpolynomial dimension.*

Proof. Let $(A_i)_{i \in [n]} \cup (B_j)_{j \in [n]} \cup (C_{(i,j)})_{(i,j) \in [n]^2}$ be a spectrahedral representation of the hyperbolicity cone $\Lambda_+(\mu, \mathbf{e})$ of the polynomial $\mu(\mathbf{x}, \mathbf{w})$, where $A_i, B_j, C_{(i,j)} \in Sym_d(\mathbb{R})$ are real symmetric matrices of dimension d such that $\sum_{i \in [n]} A_i + \sum_{j \in [n]} B_j \succ 0$. Let

$$g(\mathbf{x}, \mathbf{w}) = \mathrm{Det}\left(\sum_{i,j=1}^{n} A_i x_i + B_j x_{n+j} + C_{(i,j)} w_{(i,j)} \right)$$

The irreducibility of $\mu(\mathbf{x}, \mathbf{w})$ proved in Lemma 4.2, together with Proposition 2.5 tell us that $\mu(\mathbf{x}, \mathbf{w})$ divides $g(\mathbf{x}, \mathbf{w})$. If $d = \mathrm{poly}(n)$, the equality above gives an arithmetic circuit of size $\mathrm{poly}(d)$ computing $g(\mathbf{x}, \mathbf{w})$. In this case Proposition 3.5 and $\mu(\mathbf{x}, \mathbf{w}) \mid g(\mathbf{x}, \mathbf{w})$ imply that $\mu(\mathbf{x}, \mathbf{w})$ is computed by algebraic circuits of polynomial size, which contradicts Lemma 4.1. □

5 CONCLUSION AND OPEN PROBLEMS

In this paper we gave the first (conditional) lower bound on the spectrahedral representation of an *explicit* hyperbolicity cone which is known to be spectrahedral. An important component of our proof was to observe that the algebraic circuit complexity of the minimal defining polynomial of this hyperbolicity cone plays an important role in lower bounding the spectrahedral representation. Removing the standard complexity assumption on the proof above is the first open problem left by this work. It would be interesting to see whether the hyperbolicity assumption, and the special nature of the spectrahedral (or definite determinantal) representation could be further used to improve the lower bound above.

Another interesting question, in the viewpoint of optimization, is whether the complexity of representing a hyperbolicity cone (the ones known to be spectrahedral) via its hyperbolic polynomial can in general be much more efficient than representing it via its spectrahedral representation. This could show that using hyperbolic polynomials could provide faster ways of testing membership in in the hyperbolicity cone, than via checking the corresponding inequality given by the spectrahedral representation.

To achieve such a separation between representation by giving a circuit for the hyperbolic polynomial, one would have to find a hyperbolic polynomial (with a spectrahedral hyperbolicity cone) which can be computed by small algebraic circuits, but any definite determinantal representation of it is large. The elementary symmetric polynomials are great candidates for such separation, as they can be computed by algebraic circuits of $O(n^3)$ size. On the other hand, the best upper bound on the spectrahedral representation of the hyperbolicity cones of the elementary symmetric polynomials is exponential [1, 2]. Thus, another open question is to obtain a lower bound on the spectrahedral representation of these hyperbolicity cones.

For optimization, the best possible separation which could show the advantages of hyperbolic programming is with respect to spectrahedral shadows. In this case, one would have to exhibit a hyperbolicity cone which can be efficiently described through a small algebraic circuit computing its minimal defining polynomial, but for which any spectrahedral shadow of this cone is of superpolynomial size.

ACKNOWLEDGMENTS

The author is grateful to Nikhil Srivastava for posing the question at the Simons program and for useful conversations throughout this work. The author is also grateful to Mario Kummer and Rainer Sinn for useful conversations throughout the course of this work. This work started while the author was a research fellow at the Simons Institute, Berkeley and part of this work was done while the author was a postdoctoral fellow at the Department of Computer Science, University of Toronto.

REFERENCES

[1] Nima Amini. 2019. Spectrahedrality of hyperbolicity cones of multivariate matching polynomials. *Journal of Algebraic Combinatorics* 50, 2 (2019), 165–190.

[2] Petter Brändén. 2014. Hyperbolicity cones of elementary symmetric polynomials are spectrahedral. *Optimization Letters* 8, 5 (2014), 1773–1782.

[3] Lars Gårding. 1959. An inequality for hyperbolic polynomials. *Journal of Mathematics and Mechanics* (1959), 957–965.

[4] Osman Güler. 1997. Hyperbolic polynomials and interior point methods for convex programming. *Mathematics of Operations Research* 22, 2 (1997), 350–377.

[5] J William Helton and Victor Vinnikov. 2007. Linear matrix inequality representation of sets. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 60, 5 (2007), 654–674.

[6] Erich Kaltofen. 1989. Factorization of polynomials given by straight-line programs. *Randomness and Computation* 5, 375-412 (1989), 2–3.

[7] Mario Kummer. 2016. Two results on the size of spectrahedral descriptions. *SIAM Journal on Optimization* 26, 1 (2016), 589–601.

[8] Mario Kummer. 2017. Determinantal representations and Bézoutians. *Mathematische Zeitschrift* 285, 1-2 (2017), 445–459.

[9] Peter D Lax. 1958. Differential equations, difference equations and matrix theory. *Communications on Pure and Applied Mathematics* 11, 2 (1958), 175–194.

[10] Tim Netzer and Raman Sanyal. 2015. Smooth hyperbolicity cones are spectrahedral shadows. *Mathematical Programming* 153, 1 (2015), 213–221.

[11] Prasad Raghavendra, Nick Ryder, Nikhil Srivastava, and Benjamin Weitz. 2019. Exponential lower bounds on spectrahedral representations of hyperbolicity cones. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2322–2332.

[12] James Renegar. 2004. *Hyperbolic programs, and their derivative relaxations.* Technical Report. Cornell University Operations Research and Industrial Engineering.

[13] Claus Scheiderer. 2018. Spectrahedral shadows. *SIAM Journal on Applied Algebra and Geometry* 2, 1 (2018), 26–44.

[14] Amir Shpilka and Amir Yehudayoff. 2010. Arithmetic circuits: a survey of recent results and open questions. *Foundations and Trends® in Theoretical Computer Science* 5, 3–4 (2010), 207–388.

[15] Volker Strassen. 1973. Vermeidung von Divisionen. *Journal für die reine und angewandte Mathematik* 264 (1973), 184–202.

[16] Leslie G Valiant. 1979. Completeness classes in algebra. In *Proceedings of the eleventh annual ACM symposium on Theory of computing*. 249–261.

[17] Victor Vinnikov. 2012. LMI representations of convex semialgebraic sets and determinantal representations of algebraic hypersurfaces: past, present, and future. In *Mathematical methods in systems, optimization, and control*. Springer, 325–349.