Noether normalization

For today we will follow the convention of the course and assume that $\mathbb{k}$ is a field with $\mathrm{char}(\mathbb{k}) = 0$ ($\therefore$ $\mathbb{k}$ infinite).

We begin with a warm-up lemma:

Lemma 1: let $f \in \mathbb{k}[x_1, \dots, x_n]$ be a non-zero polynomial. Then, there are $\lambda \in \mathbb{k} \setminus \{0\}$ and $a_1, \dots, a_{n-1} \in \mathbb{k}$ s.t. $\lambda \cdot f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n, x_n)$ is monic in x_n.

Proof: if $d := \deg f$ we can write $f = f_d + \dots + f_0$ with $f_d \not\equiv 0$ homogeneous of degree d. Thus, we have $f_d(x_1, \dots, x_{n-1}, 1) \not\equiv 0$.
By the polynomial identity lemma, $\exists\ a_1, \dots, a_{n-1} \in \mathbb{k}$ s.t. $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Since the coefficient of x_n^d in $f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n, x_n)$ is given by $f_d(a_1, \dots, a_{n-1}, 1)$, by taking $\lambda = f_d(a_1, \dots, a_{n-1}, 1)^{-1}$ we have $\lambda f(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n, x_n)$ monic. $\square$

Theorem (Noether normalization): let R be a finitely generated $\mathbb{k}$-algebra, with generators $x_1, \dots, x_n$. Then, there are algebraically independent elements $z_1, \dots, z_d \in R$ such that R is module-finite over the subring $\mathbb{k}[z_1, \dots, z_d]$, which is isomorphic to a polynomial ring (d may be zero).

Proof: Induction on the number of generators of R (i.e. n).
<u>Base case</u>: $n = 0$. In this case, take $d = 0$ since $R = \mathbb{k}$.
<u>Inductive step</u>: assume we know the theorem for algebras generated by $n-1$ or fewer elements. If $x_1, \dots, x_n$ are algebraically independent, then take $d = n$ and we are done. Else, $\exists\ f \in \mathbb{k}[y_1, \dots, y_n] \setminus \{0\}$ s.t. $f(x_1, \dots, x_n) = 0$. Since $f \not\equiv 0$, Lemma 1 implies that $\exists\ a_1, \dots, a_{n-1} \in \mathbb{k}$ and $\lambda \in \mathbb{k} \setminus \{0\}$ s.t. $p(\underline{y}) := \lambda \cdot f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n)$ is monic in y_n.
Instead of using the generators $x_1, \dots, x_n$ of R, we could use the following generators: $x_1 - a_1 x_n, \dots, x_{n-1} - a_{n-1} x_n, x_n$ (these generate R as a $\mathbb{k}$-algebra) and by the above we have that $p(\underline{y})$ is a monic polynomial in y_n s.t. $p(x_1 - a_1 x_n, \dots, x_{n-1} - a_{n-1} x_n, x_n) =$
$= \lambda \cdot f(x_1, \dots, x_n) = 0 \Rightarrow x_n$ is integral over $\mathbb{k}[x_1, \dots, x_{n-1}] =: R'$ $\therefore$ R is module-finite over R'.
Since R' has $n-1$ generators, by induction we have that R' is module-finite over some polynomial ring $\mathbb{k}[z_1, \dots, z_d] \subset R'$ $\Rightarrow$ R is module-finite over $\mathbb{k}[z_1, \dots, z_d]$ as well. $\square$ by exercise 1.1.

Corollary 2: let R be a finitely generated $\mathbb{k}$-algebra, and suppose R is a field. Then R is an algebraic extension of $\mathbb{k}$.
Proof: By Noether normalization, R is module-finite over some polynomial subring $\mathbb{k}[z_1, \dots, z_d] =: T$.
If $d \geq 1$ then T has dimension at least 1 but $T \subset R$ and $\dim R = 0$ (as R is a field) contradiction.
Thus $d = 0$ and R is module-finite over $\mathbb{k}$. Since R is a field, this means R is algebraic over $\mathbb{k}$. $\square$