Effective Nullstellensatz

Suppose $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ with $d_i := \deg f_i$ and $d_1 \geq d_2 \geq \dots \geq d_m$.

HN (weak form): $V(f_1, \dots, f_m) = \emptyset \Leftrightarrow 1 \in (f_1, \dots, f_m)$.

Hermann's degree bound for ideal membership $\Rightarrow 1 \in (f_1, \dots, f_m)$
$\Leftrightarrow \exists\, g_1, \dots, g_m \in \mathbb{C}[x_1, \dots, x_n]$ with $\deg g_i \leq (2d_1)^{n2^n}$ s.t. $1 = f_1g_1 + \dots + f_mg_m$.

Brownawell (1987), Kollar (1988), Sombra ($\sim$1996), Jelonek (2004):
$1 \in (f_1, \dots, f_m) \Leftrightarrow \exists\, g_1, \dots, g_m$ s.t. $1 = \sum_{i=1}^{m} f_ig_i$ and

$$\deg f_ig_i \leq \begin{cases} \prod_{i=1}^{m} d_i & \text{if } m \leq n \\ \left(\prod_{i=1}^{n-1} d_i\right) d_m & \text{if } m > n \end{cases}$$

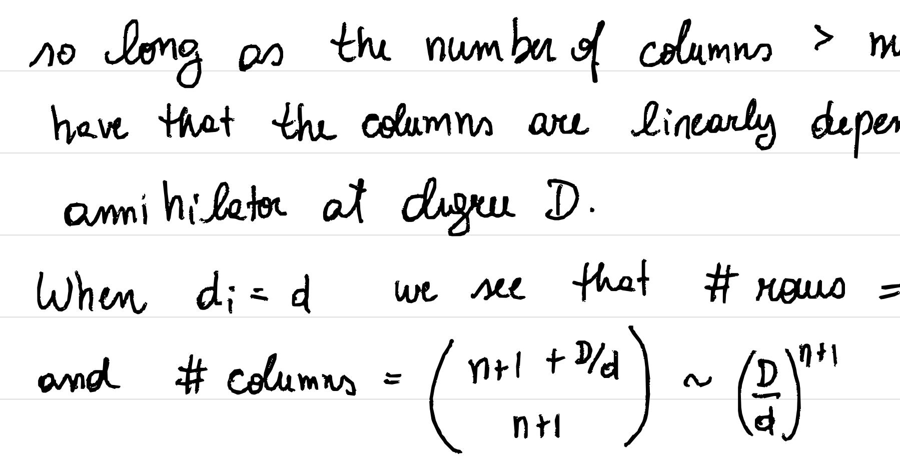
$\therefore$ HN $\in$ PSPACE.

The above bound is tight.
Example (Brownawell): $f_1 = x_1^d$, $f_2 = x_2^d - x_1$, $f_3 = x_3^d - x_2, \dots,$
$f_{n-1} = x_{n-1}^d - x_{n-2}$, $f_n = 1 - x_n^d x_{n-1}$ unsatisfiable system
$\therefore 1 = \sum_{i=1}^{n} f_ig_i \Rightarrow$ substituting $x_n = t^{-1}$, $x_{n-1} = t^d$, $x_{n-2} = t^{d^2}, \dots$
$x_1 = t^{d^{n-1}} \Rightarrow 1 = t^{d^n} \cdot g_1(t^{d^{n-1}}, \dots, t^d, t^{-1})$
$\Rightarrow \deg_{x_n} g_1 \geq d^n$.

Interlude: Algebraic Independence

$f_1, \dots, f_m$ are algebraically dependent if $\exists\, A \in \mathbb{C}[y_1, \dots, y_m] \setminus \{0\}$ s.t. $A(f_1, \dots, f_m) = 0$.
Natural question: can we bound the degree of such A?
Focus on the case $m = n+1$. In that case the set of annihilators forms a principal ideal.
Theorem (Perron 1950s): $\exists\, A \in \mathbb{C}[y_1, \dots, y_{n+1}] \setminus \{0\}$ with $A(f_1, \dots, f_{n+1}) = 0$ with w-$\deg A \leq \prod_{i=1}^{n+1} d_i$ where $\deg y_i = d_i$.

"Proof:" assume $d_1 = d_2 = \dots = d_{n+1} = d$. Let $D \in \mathbb{N}$.
Consider all monomials in $f_1, \dots, f_{n+1}$ of degree $\leq D$.
$\mathcal{M}_D := \{ f_1^{e_1} \cdots f_{n+1}^{e_{n+1}} \mid \deg(f_1^{e_1} \cdots f_{n+1}^{e_{n+1}}) \leq D \}$
Let $\Delta := \{ \bar{e} \in \mathbb{N}^{n+1} \mid \sum_{i=1}^{n+1} d_ie_i \leq D \}$.
Consider the matrix



so long as the number of columns > number of rows we have that the columns are linearly dependent $\Rightarrow$ get an annihilator at degree D.
When $d_i = d$ we see that # rows $= \binom{n+D}{n} \sim D^n$
and # columns $= \binom{n+1+D/d}{n+1} \sim \left(\frac{D}{d}\right)^{n+1}$

$\therefore$ when $D \sim d^{n+1}$ we get a dependence. $\square$

Back to HN: take $f_1, \dots, f_m$. If $m \geq n+1$ then $f_1, \dots, f_m$ are algebraically dependent $\therefore \exists\, A \neq 0$ s.t. $A(f_1, \dots, f_m) = 0$. If $A(\bar{0}) \neq 0$ then we get degree bounds, since
$0 = A(f_1, \dots, f_m) = c + \sum_{\|\bar{e}\| > 0} c_{\bar{e}} \bar{f}^{\bar{e}}$.

Geometric interpretation: $A(\bar{0}) \neq 0 \Rightarrow f_1, \dots, f_m$ not approximately satisfiable.
Def.: $f_1, \dots, f_m$ approximately satisfiable if $\forall\, \epsilon > 0\ \exists\, \bar{\alpha} \in \mathbb{C}^n$ s.t. $|f_i(\bar{\alpha})| < \epsilon$.
Fact (DSS' 2018): $A(f_1, \dots, f_m) = 0$ and $A(\bar{0}) \neq 0 \Leftrightarrow f_1, \dots, f_m$ not approximately satisfiable.
Example: $f_1 = x_1$ $f_2 = 1 - x_1x_2$ approx. satisfiable.

Effective Nullstellensatz:
Let $f_1, \dots, f_m$ s.t. $V(f_1, \dots, f_m) = \emptyset$.
By HN $\exists\, g_1, \dots, g_m$ s.t. $1 = \sum_{i=1}^{m} f_ig_i$.

Let $\varphi: \mathbb{C}^{n+1} \to \mathbb{C}^{n+m}$
$(x_1, \dots, x_n, z) \longmapsto (x_1, \dots, x_n, f_1(\bar{x})z, \dots, f_m(\bar{x})z)$
φ invertible in the image $\therefore$ it has an inverse on $\overline{\Phi(\mathbb{C}^{n+1})}$

Let $\Pi: \mathbb{C}^{n+m} \to \mathbb{C}^{n+1}$
$\bar{y} = (y_1, \dots, y_{n+m}) \longmapsto (l_i(\bar{y}))_{i=1}^{n+1}$ where $l_i(\bar{y}) = \sum_{j=1}^{n+m} \gamma_{ij} y_j$
random linear forms

Let $\Psi := \Pi \circ \varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ $\therefore$
$\Psi_i(\bar{x}, z) = \gamma_{i1}x_1 + \dots + \gamma_{in}x_n + \gamma_{i(n+1)}zf_1 + \dots + \gamma_{i(n+m)}zf_m$.

$\mathbb{C}[\Psi_1, \dots, \Psi_{n+1}] \subset \mathbb{C}[\bar{x}, z]$ and by Noether normalization $\mathbb{C}[\bar{x}, z]$ is module-finite over $\mathbb{C}[\Psi_1, \dots, \Psi_{n+1}]$ (it is in fact integral).
Let P be integral equation of z over $\mathbb{C}[\Psi_1, \dots, \Psi_{n+1}]$ of lowest degree
$\therefore P \in \mathbb{C}[y_1, \dots, y_{n+1}, t]$ monic in t and $P(\Psi_1, \dots, \Psi_{n+1}, z) = 0$.
$\therefore 0 = P(\Psi_1, \dots, \Psi_{n+1}, z) = z^D + z^{D-1}P_{D-1}(\Psi_1, \dots, \Psi_{n+1}) + \dots + zP_1(\Psi_1, \dots, \Psi_{n+1})$
$+ P_0(\Psi_1, \dots, \Psi_{n+1}) = 0$ $(*)$
coefficient of z^D in $(*)$ is:
$0 = 1 + \mathrm{coeff}_z(P_{D-1}) + \mathrm{coeff}_{z^2}(P_{D-2}) + \dots + \mathrm{coeff}_{z^D}(P_0)$
$= 1 + \sum_{i=1}^{n} f_ih_i$ since every z in P_i is multiplied by one of $f_1, \dots, f_m$
thus a bound on $\deg f_ih_i$ follows from a degree bound for $\deg P$. We will get this bound from Perron's thm.
Consider $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{C}(z)[x_1, \dots, x_n]$. They are algebraically dependent $\therefore \exists\, A \in \mathbb{C}(z)[y_1, \dots, y_{n+1}] \setminus \{0\}$ s.t. $A(\Psi_1, \dots, \Psi_{n+1}) = 0$.
By Perron we also know that $\exists\, A$ with $\deg A \leq \prod_{i=1}^{n+1} \deg \Psi_i \leq \prod d_i$

By clearing denominators we can also take $A \in \mathbb{C}[z][\bar{y}]$.
Fact: $P \mid A$ since P integral with least degree.

By the fact we have $\deg P \leq \deg A$ (the degree in the Ψ variables). This gives the desired bound.