Primary Decomposition - Theory

Let $R := \mathbb{K}[x_1, \dots, x_n]$ be our polynomial ring.

Definition 1: let $I \subseteq R$ be an ideal.

1. a prime ideal $\mathfrak{P} \subseteq R$ is an associate prime of I if $\exists\, b \in R$ s.t. $\mathfrak{P} = I : (b)$.
   $\mathrm{Ass}(I) := \{ \mathfrak{P} \subseteq R \mid \mathfrak{P} \text{ associate prime of } I \}$.

2. let $\mathfrak{P}, \mathfrak{Q} \in \mathrm{Ass}(I)$ s.t. $\mathfrak{Q} \subsetneq \mathfrak{P}$. Then $\mathfrak{P}$ is called an embedded prime of I.
   Let $\mathrm{Ass}(I, \mathfrak{P}) := \{ \mathfrak{Q} \in \mathrm{Ass}(I) \mid \mathfrak{Q} \subseteq \mathfrak{P} \}$.
   The set of minimal primes is given by $\min\mathrm{Ass}(I) := \{ \mathfrak{P} \in \mathrm{Ass}(I) \mid \mathrm{Ass}(I, \mathfrak{P}) = \{\mathfrak{P}\} \}$.

3. I is called equidimensional or pure dimensional if all associate primes of I have same dimension.

4. I is a <u>primary ideal</u> if $\forall\, a, b \in R$ s.t. $ab \in I$ and $a \notin I$ then $b \in \sqrt{I}$. Let $\mathfrak{P}$ be a prime ideal, we say that I is $\mathfrak{P}$-primary if $\sqrt{I} = \mathfrak{P}$.

5. A <u>primary decomposition</u> of I is a decomposition of the form $I = Q_1 \cap Q_2 \cap \dots \cap Q_s$, where Q_i is primary for $i \in [s]$. Such a decomposition is called <u>irredundant</u> if
   - 5.1 no Q_i can be omitted
   - 5.2 $\sqrt{Q_i} \neq \sqrt{Q_j}$ $\forall\, i \neq j$.

Example: $R = \mathbb{K}[x, y]$ $I = (x^2, xy)$
Note that $\mathrm{Ass}(I) = \{ (x), (x, y) \}$ since $(x, y) = I : (x)$ and $(x) = I : (y)$ (and we can show these are the only ones). Hence (x) is a minimal prime and (x, y) is an embedded prime.
Moreover $I = (x) \cap (x, y)^2 = (x) \cap (x^2, y)$.

Lemma 2: the radical of a primary ideal is prime.

Lemma 3: let $\mathfrak{P} \subset R$ be a prime, $Q \subset R$ be a $\mathfrak{P}$-primary ideal.

1. if Q' is $\mathfrak{P}$-primary, then $Q \cap Q'$ is $\mathfrak{P}$-primary.
2. let $b \in R \setminus Q$. Then $Q : (b)$ is $\mathfrak{P}$-primary. Moreover $b \in \mathfrak{P} \iff Q \neq Q : (b)$.
3. let $\mathfrak{P}' \supset Q$ be a prime ideal, then $Q R_{\mathfrak{P}'} \cap R = Q$
4. there is $b \in R$ s.t. $\mathfrak{P} = Q : (b)$ $\therefore$ $\mathfrak{P} \in \mathrm{Ass}(Q)$.

Proof:

1. $ab \in Q \cap Q'$ and $a \notin Q \cap Q'$ $\Rightarrow$ w.l.o.g. $a \notin Q$ $\underset{Q\ \mathfrak{P}\text{-primary}}{\Rightarrow}$ $b \in \sqrt{Q} = \mathfrak{P}$ (by Lemma 1).
   Since $\sqrt{Q \cap Q'} = \sqrt{Q} \cap \sqrt{Q'} = \mathfrak{P}$ we have $Q \cap Q'$ is $\mathfrak{P}$-primary.

2. If $b \notin \mathfrak{P}$ note that $Q : (b) = Q$, since $a \in Q : (b) \Rightarrow ab \in Q \underset{b \notin \mathfrak{P} = \sqrt{Q}}{\Rightarrow} a \in Q$. Thus we can assume $b \in \mathfrak{P}$ for the first claim of this part.
   $b \in \mathfrak{P} \setminus Q \Rightarrow \exists\, n \geq 1$ s.t. $b^n \in Q$, but $b^{n-1} \notin Q$ since $\mathfrak{P} = \sqrt{Q}$.
   Hence, we have that $Q : (b) \neq Q$, since $b^{n-1} \in Q : (b)$ but $b^{n-1} \notin Q$. (This already proves the moreover part).
   Let $xy \in Q : (b)$ and $x \notin Q : (b)$. Then $bxy \in Q$ and $bx \notin Q$ $\therefore$ $y \in \sqrt{Q} \subset \sqrt{Q : (b)}$ $\therefore$ $Q : (b)$ is primary. Since $\sqrt{Q : (b)} \supseteq \sqrt{Q} = \mathfrak{P}$ to show that $Q : (b)$ is $\mathfrak{P}$-primary we only need to show that $\mathfrak{P} \supset \sqrt{Q : (b)}$. Let $x \in \sqrt{Q : (b)}$ $\therefore$ $\exists\, m \in \mathbb{N}$ s.t. $bx^m \in Q$ and $b \notin Q$ $\therefore$ $x^m \in \sqrt{Q} = \mathfrak{P} \underset{\mathfrak{P}\text{ prime}}{\Rightarrow} x \in \mathfrak{P}$.

3. Since $Q \subseteq Q \cdot R_{\mathfrak{P}'}$ and $Q \subset R$ we only need to show that $Q R_{\mathfrak{P}'} \cap R \subseteq Q$. Let $x \in Q R_{\mathfrak{P}'} \cap R$ $\therefore$ $\exists\, s \in R \setminus \mathfrak{P}'$ s.t. $sx \in R$ and $sx \in Q \underset{\substack{Q\ \mathfrak{P}\text{-primary} \\ s \notin \mathfrak{P}}}{\Rightarrow} x \in Q$.

4. Consider the following procedure: set $b_1 = 1$, while $Q : (b_t) \neq \mathfrak{P}$ then take $g_t \in \mathfrak{P} \setminus Q : (b_t)$ and update $b_{t+1} := b_t \cdot g_t$.
   We have $Q = Q : (b_1) \subsetneq Q : (b_2) \subsetneq Q : (b_3) \subsetneq \cdots \subseteq \mathfrak{P}$, by part 2. Since R is noetherian, the chain above must stabilize $\therefore$ $\exists\, t$ s.t. $Q : (b_t) = \mathfrak{P}$ (since the chain only stabilizes when $\mathfrak{P} \subseteq Q : (b_t)$). $\square$

Theorem 4: Let $I \subsetneq R$ be an ideal, then there is an irredundant primary decomposition $I = Q_1 \cap Q_2 \cap \dots \cap Q_s$.

Proof: By Lemma 3.1, enough to show that every proper ideal is the intersection of finitely many primary ideals. Suppose this is not true, and let $\mathcal{B}$ be the set of proper ideals which are not a finite intersection of primary ideals.
R noetherian $\Rightarrow$ $\mathcal{B}$ has a maximal element w.r.t. inclusion. Let J be such element. J not primary $\Rightarrow \exists\, a, b \in R$ s.t. $ab \in J$ but $a \notin J$ and $b \notin \sqrt{J}$. Consider the chain $J \subset J : (b) \subset J : (b^2) \subset \cdots$
R noetherian $\Rightarrow \exists\, n \in \mathbb{N}$ s.t. $J : (b^n) = J : (b^{n+1}) = \cdots$
Since $J = (J : (b^n)) \cap (J, b^n)$ and since $J \neq J : (b^n)$ as $a \in J : (b^n) \setminus J$ and $J \neq (J, b^n)$ as $b \notin \sqrt{J}$ by maximality of J we have $J : (b^n), (J, b^n) \notin \mathcal{B}$ $\therefore$ they are an intersection of finitely many primary ideals $\Rightarrow J \notin \mathcal{B}$ which is a contradiction. $\square$

Theorem 5: Let $I \subset R$ be a proper ideal with irredundant primary decomposition $I = Q_1 \cap Q_2 \cap \dots \cap Q_r$. Then $r = |\mathrm{Ass}(I)|$ and $\mathrm{Ass}(I) = \{ \sqrt{Q_1}, \dots, \sqrt{Q_r} \}$. Moreover, if for some $\mathfrak{P} \in \mathrm{Ass}(I)$ $\mathrm{Ass}(I, \mathfrak{P}) = \{ \sqrt{Q_{i_1}}, \dots, \sqrt{Q_{i_s}} \}$ then $Q_{i_1} \cap \dots \cap Q_{i_s}$ is independent of the decomposition.

Proof: $\mathfrak{P} \in \mathrm{Ass}(I) \Rightarrow \mathfrak{P} = I : (b)$ for some $b \in R \Rightarrow$
$\mathfrak{P} = \bigcap_{i=1}^{r} (Q_i : (b)) \underset{\mathfrak{P}\text{ prime}}{\Rightarrow} \exists\, k \in [r]$ s.t. $\mathfrak{P} = Q_k : (b)$. By Lemma 3.4 we have $\mathfrak{P} = \sqrt{Q_k}$.