Dimension of an ideal and its computation.

Definition (dimension): for a prime ideal $P \subset \mathbb{K}[\bar{x}]$
$\dim P := \operatorname{trdeg}_{\mathbb{K}} \left(\mathbb{K}[\bar{x}]/P \right)$.
For an arbitrary ideal $I \subset \mathbb{K}[\bar{x}]$, we have
$\dim I := \max_{P \in \mathrm{Ass}(I)} \dim P$. When $I = (1)$ then $\dim I := -1$.

Another way to compute the dimension is as follows:
$\dim I := \max\{ r \mid x_{i_1}, \dots, x_{i_r}$ are algebraically independent over $\mathbb{K}$ in $\mathbb{K}[\bar{x}]/I \} = \max\{ r \mid \mathbb{K}[x_{i_1}, \dots, x_{i_r}] \cap I = 0 \}$.
[Greuel-Pfister 08, Theorem 3.5.1]

Proposition 6: Given $I := (f_1, \dots, f_s) \subset \mathbb{K}[\bar{x}]$ one can decide whether $x_1, \dots, x_r$ are algebraically independent over $\mathbb{K}$ in $\mathbb{K}[\bar{x}]/I$. Moreover, if they are dependent, one can find $h \in \mathbb{K}[x_1, \dots, x_r] \cap I \setminus \{0\}$ with $\deg h \le d \cdot \binom{\lambda(n-r,1,d)+d+n-r}{n-r}$

Proof: $x_1, \dots, x_r$ are dependent $\iff I \cap \mathbb{K}[x_1, \dots, x_r] \neq 0$
$\iff 1 \in I \cdot \mathbb{K}(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$.
By proposition 5, $1 \in I \cdot \mathbb{K}(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ iff
$\exists\, g_1, \dots, g_s \in \mathbb{K}(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$ with $\deg g_i \le \lambda(n-r,1,d)$
s.t. $1 = \sum_{i=1}^{s} f_i g_i$. In particular, this implies

that the system $M\vec{g} = v_1$ has a solution, where

$M \in \mathbb{K}[x_1, \dots, x_r]_{\le d}^{N_1 \times N_2}$ where each entry M_{ij} is a "coefficient" of some f_k (when seen in $\mathbb{K}[x_1, \dots, x_r][x_{r+1}, \dots, x_n]$), hence the bound on the degree.
Moreover, the bound on $\deg g_i \Rightarrow$

$N_1 \le \binom{\lambda(n-r,1,d)+d+n-r}{n-r}$ (max degree of RHS of system)

and $N_2 \le s \cdot \binom{\lambda(n-r,1,d)+n-r}{n-r}$ (# coefficients of g_i's)

$\Rightarrow$ there is $h \in \mathbb{K}[x_1, \dots, x_r] \setminus 0$ with $\deg h \le d \cdot N_1$
generalized Cramer's rule
s.t. a solution $(g_1, \dots, g_s)$ exists with common denominator h.
This implies that $h \in \mathbb{K}[x_1, \dots, x_r] \cap I$. □

Corollary 6: given an ideal $I := (f_1, \dots, f_s) \subset \mathbb{K}[x_1, \dots, x_n]$ one can compute $\dim I$ in EXPSPACE.

Proof: by proposition 6, together with the fact that the collection of sets of algebraically independent variables forms a matroid [Roman '08, Theorem 4.2.1] gives us a way to find a maximal set of algebraically independent variables in EXPSPACE. Thus we can compute this cardinality, which gives us $\dim I$.
Note: since we are running in EXPSPACE, we don't need the matroid observation (in theory), but this helps in practice.

Proposition 7: If $I := (f_1, \dots, f_s) \subset \mathbb{K}[x_1, \dots, x_n]$ ideal is s.t. $\dim I = 0$ then we have $\dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/I$ (as a $\mathbb{K}$-vector space) is upper bounded by

$$\dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/I \le \left(d \cdot \binom{\lambda(n-1,1,d)+d+n-1}{n-1} \right)^n.$$

Proof: $\dim I = 0 \Rightarrow$ every x_i is algebraic over $\mathbb{K}$ in $\mathbb{K}[x_1, \dots, x_n]/I \underset{\text{proposition 6}}{\Rightarrow} \exists\, h_i \in \mathbb{K}[x_i] \cap I \setminus 0$ with

$\deg h_i \le d \cdot \binom{\lambda(n-1,1,d)+d+n-1}{n-1}$ $\therefore$

$$\dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/I \le \dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/(h_1, \dots, h_n) = \prod \deg h_i \quad \square$$