Fundamental subroutine: linear system of homogeneous polynomial equations

Proposition 1: given system of homogeneous equations $\sum_{j=1}^{s} f_{ij} g_j = 0 \quad i \in [r]$, $f_{ij} \in \mathbb{K}[\bar{x}]$, one can construct a $\mathbb{K}[\bar{x}]$-module basis for the solutions $(g_1, \dots, g_s) \in \mathbb{K}[\bar{x}]^s$.
Moreover there is a bound $\lambda(n,r,d) \leq 2 \cdot (2dr)^{2^n}$, such that a basis of solutions exists with $\deg g_i \leq \lambda(n,r,d)$.
In particular this yields a bound $\Lambda(n,r,d)$ on the number of operations (in $\mathbb{K}$) needed to construct such basis.

Proof: We can assume $\mathbb{K}$ infinite, otherwise let u be new variable and suppose $(g_i(u,\bar{x}))_{i=1}^{s}$ is a solution over $\mathbb{K}(u)[\bar{x}]$.
Clearing denominators we can assume $g_i(u,\bar{x}) \in \mathbb{K}[u,\bar{x}]$ now, by taking the coefficients of any power of u we get a solution over $\mathbb{K}$.
Hence from a basis of solutions over $\mathbb{K}(u)$ one gets a basis of solutions over $\mathbb{K}$.

Let $M := (f_{ij})_{i \in [r], j \in [s]}$. Since $\operatorname{rank}_{\mathbb{K}(\bar{x})} M = r$, w.l.o.g.

$N := (f_{ij})_{i,j=1}^{r}$ is nonsingular (over $\mathbb{K}(\bar{x})$) and let

$\Delta := \det N$. ($\therefore \Delta \neq 0$)
Since $\mathbb{K}$ is infinite, after an appropriate linear transformation we can assume Δ monic in X_n (i.e. regular in $\bar{x}$).
Let $\tilde{N} := \operatorname{adj}(N)$ $\therefore$ $\tilde{N} N = \Delta \cdot I_r$ and we have

$$M \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix} = 0 \iff 0 = \tilde{N} M \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix} = \left(\Delta I_r \mid -Q\right) \begin{pmatrix} g_1 \\ \vdots \\ g_s \end{pmatrix}$$

$$\iff \Delta g_i = Q_{i(r+1)} g_{r+1} + \dots + Q_{is} g_s \qquad i \in [r]$$

where $Q \in \mathbb{K}[\bar{x}]^{r \times (s-r)}$.
We have the following set of "special" solutions:
$(p_{i1}, \dots, p_{ir}, \underbrace{0, \dots, 0}_{i-1}, \Delta, 0, \dots, 0) =: v_i$.

In particular, for any solution $(a_1, \dots, a_s)$, by monic division by Δ we can write
$(a_1, \dots, a_s) = (a_1', \dots, a_s') + \sum_{i=1}^{s-r} t_i \cdot v_i$ where $t_i \in \mathbb{K}[\bar{x}]$

and $\deg_n a_i' < \deg_n \Delta$ for $r < i \leq s$.
So we can construct a basis by starting with $v_1, \dots, v_{r-s}$ and then adding solutions $(a_1, \dots, a_s)$ with $\deg_n a_i < \deg_n \Delta$ for $r < i$ (this also bounds $\deg_n a_i < \underbrace{\deg_n \Delta + d}_{D_n + 1}$ for $i \in [r]$).

In particular, these solutions can be found by analyzing the following system in $n-1$ variables (i.e. $(x_1, \dots, x_{n-1})$):
write $f_{ij} = \sum_{k=0}^{d} f_{ijk} x_n^k \qquad g_i = \sum_{k=0}^{D_n} \underbrace{g_{ik}}\, x_n^k$

these are the new unknowns

and our system becomes (collecting powers of x_n):

$$\sum_{j=1}^{s} \sum_{a=0}^{k} f_{ija}\, g_{j(k-a)} = 0 \qquad i \in [r],\ 0 \leq k \leq D_n$$

which has $\underbrace{r \cdot (D_n + 1)}_{\tilde{r}}$ equations in $\underbrace{s \cdot D_n}_{\tilde{s}}$ unknowns.

Since $D_n + 1 := \deg_n \Delta + d \leq 2rd$, we have

$$\lambda(n,r,d) \leq 2dr + \lambda(n-1, 2dr^2, d)$$

in particular, if we define by r_k the upper bound on the rank of the linear system when we have k variables we have the following recurrence:
$r_{n-1} \leq 2d r_n^2 \iff 2d r_{n-1} \leq (2d r_n)^2$
$\therefore$ the general recurrence is given by $2d r_{n-k-1} \leq (2d r_{n-k})^2$
$\Rightarrow 2d r_{n-k} \leq (2d r_n)^{2^k} = (2dr)^{2^k}$
$\Rightarrow \lambda(n,r,d) \leq \sum_{j=0}^{n} 2d r_{n-j} \leq 2 \cdot (2dr)^{2^n}$.

To get the recurrence just note by the above argument that $r_{n-k-1} \leq r_{n-k} \cdot D_{n-k}$ and $D_{n-k} \leq 2d \cdot r_{n-k}$. $\square$