Finiteness Conditions

Let $R \subset S$ be rings.

Module-finiteness: S is module-finite over R if S is finitely generated as an R-module.

Ring-finiteness: S is ring-finite over R if $\exists\, m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ s.t. $S = R[s_1, \ldots, s_m]$
Also known as finitely generated R-algebra.

Field extension: if R, S are fields then S is a field extension of R if $\exists\, m \in \mathbb{N}$ and $s_1, \ldots, s_m \in S$ such that $S = R(s_1, \ldots, m)$.

Exercise 1: let $R \subset S \subset T$ be rings

① if $S = \sum_{i=1}^{m} R \cdot s_i$ and $T = \sum_{i=1}^{n} S \cdot t_i$ then $T = \sum_{i=1}^{m} \sum_{j=1}^{n} R s_i t_j$

② if $S = R[x_1, \ldots, x_m]$ and $T = S[y_1, \ldots, y_n]$ then $T = R[\bar{x}, \bar{y}]$

③ if R, S, T are fields and $S = R(x_1, \ldots, x_m)$, $T = S(y_1, \ldots, y_n)$ then $T = R(\bar{x}, \bar{y})$.

Thus each finiteness condition is transitive.

Integral elements

Let again $R \subset S$ be rings. An element $s \in S$ is said to be integral over R if there is a monic polynomial $F \in R[x]$ s.t. $F(v) = 0$.

We say that S is integral over R if every $s \in S$ is integral over R.

When R and S are fields and S is integral over R, we say that S is an algebraic extension of R.

Algebraic Independence

Let S be an R-algebra and $z_1, \ldots, z_d \in S$.
We say that elements $z_1, \ldots, z_d$ are algebraically independent over R if the R-algebra homomorphism $\varphi : R[x_1, \ldots, x_d] \longrightarrow R[z_1, \ldots, z_d]$ defined by $\varphi(x_i) = z_i$ is an isomorphism (where $R[x_1, \ldots, x_d]$ is the polynomial ring).
The failure of $z_1, \ldots, z_d$ being algebraically independent is equivalent to $\ker(\varphi) \neq 0$, i.e., $\exists\, f \in R[x_1, \ldots, x_d] \setminus \{0\}$ s.t. $f(z_1, \ldots, z_d) = 0$.