Lecture 20: Matrix Multiplication & Exponent of Linear Algebra

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Overview

- Matrix Multiplication
- The Exponent of Linear Algebra
- Matrix Inversion
- Determinant and Matrix Inverse
- Computing Partial Derivatives
- Conclusion

Matrix Multiplication

- **Input:** matrices $A, B \in \mathbb{F}^{n \times n}$
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  Can we do better?
- Strassen 1969: YES!
- Idea: divide matrix into blocks, and *reduce number of multiplications* needed!

Strassen's Algorithm

- Suppose that $n = 2^k$
- Let $A, B, C \in \mathbb{F}^{n \times n}$ such that $C = AB$. Divide them into blocks of size $n/2$:

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$$P_1 = A_{11}B_{11},\ P_2 = A_{12}B_{21},\ P_3 = S_4B_{22},\ P_4 = A_{22}T_4$$

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- Correctness follows from the computations

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- Could also use Master theorem to get $MM(n) = O(n^{\log 7}) \approx O(n^{2.807})$

Matrix Multiplication Exponent

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  1. If an algebraic algorithm for $n \times n$ matrix multiplication uses $O(n^{\alpha})$ operations, then $\omega \leq \alpha$.
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- As we will see today, ω is a fundamental constant in computer science!
- Currently we know $2 \leq \omega < 2.376$

Open Question

What is the right value of ω?

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- Motivated work on better algorithms for all other linear algebraic problems
- introduced complexity of computation of *bilinear functions* and the study of complexity of tensor decompositions

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- More generally, all of these ω_P's are related to ω!

  Matrix multiplication exponent fundamental to linear algebra!

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- How to prove this? *reductions!*

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- So if we could invert in time T, then we can multiply two matrices in time $O(T)$.

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- How do we compute this? *Schur Complement*

  Similar to how we would invert regular matrices! Just pay attention to non-commutativity.

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- Recurrence relation:

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- Thus

$$\begin{aligned} I(n) = I(2^k) &\leq 2^k \cdot I(1) + C \cdot \sum_{j=0}^{k-1} 2^{\omega j} \\ &\leq C' \cdot \left(2^k + \frac{2^{\omega k} - 1}{2^{\omega} - 1}\right) \\ &\leq C'' \cdot 2^{\omega k} = C'' n^{\omega} \end{aligned}$$

Determinant vs Matrix Multiplication

- One can similarly prove that $\omega_{determinant} \leq \omega$
- This is your homework! :)

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Determinant of a Matrix

- Given matrix $M \in \mathbb{F}^{n \times n}$, the determinant is

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- Entries of the adjugate (determinants of minors) are very much related to *derivatives* of the determinant of M

$$\det(M^{(i,j)}) = (-1)^{i+j} \partial_{i,j} \det(M)$$

- So, if we knew how to compute the determinant AND ALL its partial derivatives, we could:
  1. Compute the adjugate
  2. Compute the inverse

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- Compute the inverse by simply dividing $\det(M^{(i,j)})/\det(M)$

- Matrix Multiplication
- The Exponent of Linear Algebra
- Matrix Inversion
- Determinant and Matrix Inverse
- Computing Partial Derivatives
- Conclusion

Algebraic Circuits - base ring R

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- *circuit size:* number of edges in the circuit, denoted by $\mathcal{S}(\Phi)$

Computing Partial Derivatives

Theorem

If $f(x_1, \ldots, x_n)$ can be computed by an algebraic circuit of size $\leq L$ then there is an algebraic circuit of size $\leq 4L$ that computes ALL partial derivatives $\partial_1 f, \ldots, \partial_n f$ simultaneously!

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- Algorithm we will see today discovered independently in Machine Learning - known as *backpropagation*

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- We are going to use the chain rule:

$$\partial_i f(g_1, g_2, \ldots, g_m) = \sum_{j=1}^{m} (\partial_j f)(g_1, g_2, \ldots, g_m) \cdot \partial_i g_j$$

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  3. Have to compute partial derivatives "*in reverse*"

Conclusion

- Today we learned how fundamental matrix multiplication is in symbolic computation and linear algebra
- Used fast computation of partial derivatives to compute the inverse from the determinant