Lecture 17: Online Algorithms & Paging

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Overview

- Part I
  - Why Study Online Algorithms?
  - Competitive Analysis
  - Examples
- Paging & Caching
- Conclusion
- Acknowledgements

Why Study Online Algorithms?

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.

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  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...

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  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future* (that is, see the entire input beforehand)[1]
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- Today, we will only see algorithms which must deal with the input as it receives it, *no constraints in memory*.
  1. Goal here is to *be competitive* against *any offline algorithm* (that is, algorithms that could see the entire input beforehand)
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Definition (Deterministic Competitive Ratio)

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- Each time we go skiing, we have to decide whether to buy or rent (unless we bought it beforehand)
- Algorithm has to decide *when to buy*, knowing only that we have gone skiing t times

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  - If $t \geq 10$, we buy at the 10^{th} time, so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot 9 + 1000}{1000} = 1.9$$

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- **Goal:** maximize probability of dating the best person

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- If $\pi(k) = 1$, then $1 - P_k$ is the probability that we picked a person between $[t+1, k-1]$, which means someone in this range better than the first t people.

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- Optimizing we get that we should set $t = n/e$, which gives us $1/e$ probability.
- Wait a second, where is the competitive analysis?

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- Complicated algorithm, based on computing time steps $t_0 \leq t_1 \leq \dots$ and between timesteps t_k and t_{k+1} we are willing to pick person who is $\leq k+1$ best from our current list.

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  - With constant probability, rank of the last person is $\Omega(n)$, so we either date the best, or we date someone in the "bottom percentile" of our list
  - Expected rank of our life-long partner is $\Omega(n)$
- Can we do better?
- Yes! There is an algorithm that picks person of average rank $O(1)$, which is therefore $O(1)$-competitive.
- Complicated algorithm, based on computing time steps $t_0 \leq t_1 \leq \dots$ and between timesteps t_k and t_{k+1} we are willing to pick person who is $\leq k+1$ best from our current list.
- That is, as we get older, we become more desperate to find someone and lower our expectations...

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- Simplification: assume we only have cache and main memory.

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1. **Least Recently Used (LRU)**: k-competitive
2. **Random**: k-competitive
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LRU Analysis

Theorem

For cache of size k, LRU is k-competitive.

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1. Upper bound: divide input sequence into phases.
   - First phase starts immediately after our algorithm first faults, ends right after the algorithm faults k more times
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   - and so on...

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4. Examples of phases, for $k = 3$:

   1, 1, 2, 2, 1, 3, 4, 3, 2, 4, 5, 6, 15, 4, 4, 2, 3, 5, 6, 4,5

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LRU Analysis - Example

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1 (1, 2, 2, 1, 3, 4) (3, 2, 4, 5, 6) (15, 4, 4, 2) (3, 5, 6) (4, 5

LRU Analysis - Upper Bound

- Need to prove that OPT will fault at least once per phase.
- If the same page faulted twice in one phase:

LRU Analysis - Upper Bound

- If each page faulted once in a phase.

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- **Claim:** in the beginning of each phase, content of OPT and content of our algorithm A intersect in at least one page.
- Proof: Look at last fault page in previous phase.

LRU Analysis - Upper Bound

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- **Claim:** in the beginning of each phase, content of OPT and content of our algorithm A intersect in at least one page.
- Since OPT and A had a common page, then OPT must have faulted as well (since each page faulted in this phase)

Lower Bound - Deterministic Paging Algorithms

Theorem

Any deterministic algorithm for paging with k pages is at least k-competitive!

- Proof by trolling.[6] Let's use $k+1$ pages, and let A be our paging algorithm.

[6] Common lower bound technique for online algorithms, also commonly used online as well :)

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- When offline algorithm deletes a page, it's next delete happens after at least k steps.

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Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*

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- Lecture based largely on:
  - Lecture 17 of Luca's Optimization class
  - Lectures 19 and 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Luca's Lecture 17 notes at

  `https://lucatrevisan.github.io/teaching/cs261-11/lecture17.pdf`
- See Karger's Lecture 19 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s22-online.pdf`
- See Karger's Lecture 20 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf`

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