

Lecture 14: Positive Semidefinite Matrices & Semidefinite Programming

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

June 12, 2024

Overview

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- Application: Control Theory
- Conclusion
- Acknowledgements

Symmetric Matrices & Spectral Theorem

- A matrix $S \in \text{Mat}(n, \mathbb{R})$ is symmetric if $S_{ij} = S_{ji}$ for all $i, j \in [n]$.

Symmetric Matrices & Spectral Theorem

- A matrix $S \in \text{Mat}(n, \mathbb{R})$ is symmetric if $S_{ij} = S_{ji}$ for all $i, j \in [n]$.
- $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if there exists $u \in \mathbb{C}^n$ such that $Su = \lambda u$.
The vector u is an *eigenvector* of S corresponding to λ .

Symmetric Matrices & Spectral Theorem

- A matrix $S \in \text{Mat}(n, \mathbb{R})$ is symmetric if $S_{ij} = S_{ji}$ for all $i, j \in [n]$.
- $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if there exists $u \in \mathbb{C}^n$ such that $Su = \lambda u$. The vector u is an *eigenvector* of S corresponding to λ .
- **Spectral theorem:** any symmetric matrix in $\text{Mat}(n, \mathbb{R})$ has n real eigenvalues (counting with multiplicity), as well as an orthonormal basis (in $\mathbb{R}^n$) for the eigenvectors.

Symmetric Matrices & Spectral Theorem

- A matrix $S \in \text{Mat}(n, \mathbb{R})$ is symmetric if $S_{ij} = S_{ji}$ for all $i, j \in [n]$.
- $\lambda \in \mathbb{C}$ is an **eigenvalue** of S if there exists $u \in \mathbb{C}^n$ such that $Su = \lambda u$. The vector u is an **eigenvector** of S corresponding to λ .
- **Spectral theorem:** any symmetric matrix in $\text{Mat}(n, \mathbb{R})$ has n real eigenvalues (counting with multiplicity), as well as an orthonormal basis (in $\mathbb{R}^n$) for the eigenvectors.
- In other words, we can write

$$S = \sum_{i=1}^n \lambda_i u_i u_i^T$$

where $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$ such that $\langle u_i, u_j \rangle = \delta_{ij}$.

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative
 - 2 $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative
 - 2 $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - 3 $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - ① all eigenvalues of S are non-negative
 - ② $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - ③ $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$
 - ④ $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative
 - 2 $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - 3 $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$
 - 4 $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular
 - 5 S is in the convex hull of the set

$$\{uu^T \mid u \in \mathbb{R}^n\}$$

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - ① all eigenvalues of S are non-negative
 - ② $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - ③ $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$
 - ④ $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular
 - ⑤ S is in the convex hull of the set

$$\{uu^T \mid u \in \mathbb{R}^n\}$$

- ⑥ $S = U^T D U$, where D is diagonal and non-negative and $U \in \text{Mat}(n, \mathbb{R})$ is orthonormal matrix (that is, $U^T U = I$).

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative
 - 2 $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - 3 $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$
 - 4 $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular
 - 5 S is in the convex hull of the set

$$\{uu^T \mid u \in \mathbb{R}^n\}$$

- 6 $S = U^T D U$, where D is diagonal and non-negative and $U \in \text{Mat}(n, \mathbb{R})$ is orthonormal matrix (that is, $U^T U = I$).
- 7 Any principal submatrix of S has non-negative determinant

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \text{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.
- There are several equivalent characterizations of PSD matrices:
 - 1 all eigenvalues of S are non-negative
 - 2 $S = Y^T Y$ for some $Y \in \mathbb{R}^{d \times n}$, where $d \leq n$
 - 3 $x^T S x \geq 0$ for all $x \in \mathbb{R}^n$
 - 4 $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular
 - 5 S is in the convex hull of the set

$$\{uu^T \mid u \in \mathbb{R}^n\}$$

- 6 $S = U^T D U$, where D is diagonal and non-negative and $U \in \text{Mat}(n, \mathbb{R})$ is orthonormal matrix (that is, $U^T U = I$).
 - 7 Any principal submatrix of S has non-negative determinant
- **Practice problem:** prove that these are all equivalent!

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- Application: Control Theory
- Conclusion
- Acknowledgements

Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n\end{array}$$

Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n\end{array}$$

- Very general family of problems.

Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n\end{array}$$

- Very general family of problems.
- Special case when all $f, g_1, \dots, g_m$ are *linear*. *Linear Programming*

Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n\end{array}$$

- Very general family of problems.
- Special case when all $f, g_1, \dots, g_m$ are *linear*. *Linear Programming*
- More general case: *Semidefinite Programming*
 - ① $A_1, \dots, A_n, B \in \mathcal{S}^m$ are $m \times m$ symmetric matrices

Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n\end{array}$$

- Very general family of problems.
- Special case when all $f, g_1, \dots, g_m$ are *linear*. Linear Programming
- More general case: Semidefinite Programming
 - 1 $A_1, \dots, A_n, B \in \mathcal{S}^m$ are $m \times m$ symmetric matrices
 - 2 Constraints:

$$x_1 \cdot A_1 + \dots + x_n \cdot A_n \succeq B$$

- 3 Minimize linear function $c^T x$

What is a Semidefinite Program?

- $\mathcal{S}^m := \mathcal{S}^m(\mathbb{R})$ space of all $m \times m$ symmetric matrices (real entries)

What is a Semidefinite Program?

- $\mathcal{S}^m := \mathcal{S}^m(\mathbb{R})$ space of all $m \times m$ symmetric matrices (real entries)

Semidefinite Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n \\ & A_i, B \in \mathcal{S}^m(\mathbb{R}) \text{ (fixed matrices)}\end{array}$$

What is a Semidefinite Program?

- $\mathcal{S}^m := \mathcal{S}^m(\mathbb{R})$ space of all $m \times m$ symmetric matrices (real entries)

Semidefinite Programming deals with problems of the form

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n \\ & A_i, B \in \mathcal{S}^m(\mathbb{R}) \text{ (fixed matrices)}\end{array}$$

Where we use $C \succeq D$ to denote that $C - D \succeq 0$ (i.e., $C - D$ is PSD).

How does it generalize Linear Programming?

Linear Programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \in \mathbb{R}^n\end{array}$$

How does it generalize Linear Programming?

Linear Programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \in \mathbb{R}^n\end{array}$$

Semidefinite Programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

How does it generalize Linear Programming?

Linear Programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \in \mathbb{R}^n\end{array}$$

Semidefinite Programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

Set A_i 's to be diagonal matrices, and $B = \text{diag}(b_1, \dots, b_m)$

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!
- Semidefinite Programming is no different!

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!
- Semidefinite Programming is no different!
 - equilibrium analysis of dynamics and control (flight controls, robotics, etc.)
 - robust optimization
 - statistics and ML
 - continuous games
 - software verification
 - filter design
 - quantum computation and information
 - automated theorem proving
 - packing problems
 - many more

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!
- Semidefinite Programming is no different!
 - equilibrium analysis of dynamics and control (flight controls, robotics, etc.)
 - robust optimization
 - statistics and ML
 - continuous games
 - software verification
 - filter design
 - quantum computation and information
 - automated theorem proving
 - packing problems
 - many more
- See more here

<https://windowsontheory.org/2016/08/27/>

[proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/](https://windowsontheory.org/2016/08/27/proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/)

Important Questions

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

Important Questions

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

- 1 When is a Semidefinite Program *feasible*?
 - Is there a solution to the constraints at all?

Important Questions

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

- ① When is a Semidefinite Program *feasible*?
 - Is there a solution to the constraints at all?
- ② When is a Semidefinite Program *bounded*?
 - Is there a minimum? Is the minimum achievable? Or is the minimum $-\infty$?

Important Questions

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

- ① When is a Semidefinite Program *feasible*?
 - Is there a solution to the constraints at all?
- ② When is a Semidefinite Program *bounded*?
 - Is there a minimum? Is the minimum achievable? Or is the minimum $-\infty$?
- ③ Can we characterize *optimality*?
 - How can we know that we found a minimum solution?
 - Do these solutions have nice description?
 - Do the solutions have *small bit complexity*?

Important Questions

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n\end{array}$$

- ① When is a Semidefinite Program *feasible*?
 - Is there a solution to the constraints at all?
- ② When is a Semidefinite Program *bounded*?
 - Is there a minimum? Is the minimum achievable? Or is the minimum $-\infty$?
- ③ Can we characterize *optimality*?
 - How can we know that we found a minimum solution?
 - Do these solutions have nice description?
 - Do the solutions have *small bit complexity*?
- ④ How do we design *efficient algorithms* that find *optimal solutions* to Semidefinite Programs?

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- Application: Control Theory
- Conclusion
- Acknowledgements

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

Definition (Linear Matrix Inequalities)

A linear matrix inequality is an inequality of the form:

$$A_0 + \sum_{i=1}^n A_i x_i \succeq 0,$$

where $A_0, \dots, A_n$ are *symmetric matrices*.

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

Definition (Linear Matrix Inequalities)

A linear matrix inequality is an inequality of the form:

$$A_0 + \sum_{i=1}^n A_i x_i \succeq 0,$$

where $A_0, \dots, A_n$ are *symmetric matrices*.

Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}$$

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}$$

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}$$

Example of Spectrahedron

Polyhedron:

Example of Spectrahedron

Circle:

Example of Spectrahedron

Hyperbola:

Example of Spectrahedron

Elliptic curve:

Projected Spectrahedron

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

Projected Spectrahedron

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

Definition (Projected Spectrahedron)

A set $S \in \mathbb{R}^n$ is a *projected spectrahedron* if it has the form:

$$S = \left\{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^t \text{ s.t. } \sum_{i=1}^n A_i x_i + \sum_{j=1}^t B_j y_j \succeq C, \quad A_i, B_j, C \in \mathcal{S}^m \right\}$$

Projected Spectrahedron

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

Definition (Projected Spectrahedron)

A set $S \in \mathbb{R}^n$ is a *projected spectrahedron* if it has the form:

$$S = \left\{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^t \text{ s.t. } \sum_{i=1}^n A_i x_i + \sum_{j=1}^t B_j y_j \succeq C, \quad A_i, B_j, C \in \mathcal{S}^m \right\}$$

Example of Projected Spectrahedron

Projection of hyperbola:

Example of Projected Spectrahedron

Projection quadratic cone intersected with halfspace:

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}.$$

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}.$$

- Note that $x \in S$ is (by definition) equivalent to

$$Z = \sum_{i=1}^n A_i x_i - B \succeq 0$$

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}.$$

- Note that $x \in S$ is (by definition) equivalent to

$$Z = \sum_{i=1}^n A_i x_i - B \succeq 0$$

- So, how do we efficiently check if $Z \succeq 0$?

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}.$$

- Note that $x \in S$ is (by definition) equivalent to

$$Z = \sum_{i=1}^n A_i x_i - B \succeq 0$$

- So, how do we efficiently check if $Z \succeq 0$?
- Symmetric Gaussian Elimination!

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}.$$

- Note that $x \in S$ is (by definition) equivalent to

$$Z = \sum_{i=1}^n A_i x_i - B \succeq 0$$

- So, how do we efficiently check if $Z \succeq 0$?
- Symmetric Gaussian Elimination!
- We will use following characterizations of PSDness of symmetric $A \in \mathcal{S}^m$

- 1 all eigenvalues of A are *non-negative*
- 2 $A = LDL^T$ for some L lower triangular and unit diagonal, D diagonal and non-negative
- 3 $z^T A z \geq 0$ for any $z \in \mathbb{R}^m$
- 4 Any principal minor of A has non-negative determinant

How do we test membership in the Spectrahedron?

- **Input:** symmetric matrix $A \in \mathcal{S}^m$
- **Output:** YES if $A \succeq 0$, NO otherwise (and output $z \in \mathbb{R}^m$ such that $z^T A z < 0$)

How do we test membership in the Spectrahedron?

- **Input:** symmetric matrix $A \in \mathcal{S}^m$
- **Output:** YES if $A \succeq 0$, NO otherwise (and output $z \in \mathbb{R}^m$ such that $z^T A z < 0$)

- Our algorithm runs in time strongly polynomial.

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- **Application: Control Theory**
- Conclusion
- Acknowledgements

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹

¹When A non-negative and x_0 non-negative we have Markov chains.

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹
- Used to model time evolution of

¹When A non-negative and x_0 non-negative we have Markov chains.

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹
- Used to model time evolution of
 - Temperatures of objects
 - Size of population
 - Voltage of electrical circuits
 - Concentration of chemical mixtures

¹When A non-negative and x_0 non-negative we have Markov chains.

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹
- Used to model time evolution of
 - Temperatures of objects
 - Size of population
 - Voltage of electrical circuits
 - Concentration of chemical mixtures
- **Question:** when $t \rightarrow \infty$, under what conditions will $x(t)$ remain bounded? Or go to zero?

¹When A non-negative and x_0 non-negative we have Markov chains 

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹
- Used to model time evolution of
 - Temperatures of objects
 - Size of population
 - Voltage of electrical circuits
 - Concentration of chemical mixtures
- **Question:** when $t \rightarrow \infty$, under what conditions will $x(t)$ remain bounded? Or go to zero?
- When system converges to zero, we say it is *stable*.

¹When A non-negative and x_0 non-negative we have Markov chains 

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.¹
- Used to model time evolution of
 - Temperatures of objects
 - Size of population
 - Voltage of electrical circuits
 - Concentration of chemical mixtures
- **Question:** when $t \rightarrow \infty$, under what conditions will $x(t)$ remain bounded? Or go to zero?
- When system converges to zero, we say it is *stable*.
- System is stable iff $|\lambda_i(A)| < 1$

¹When A non-negative and x_0 non-negative we have Markov chains 

Stability of Linear Systems

SDP viewpoint:

- Lyapunov functions (generalize *energy* in systems). Functions on $x(t)$ decrease monotonically on trajectories of the system.

Stability of Linear Systems

SDP viewpoint:

- Lyapunov functions (generalize *energy* in systems). Functions on $x(t)$ decrease monotonically on trajectories of the system.
- For our discrete-time system, we have:

$$V(x(t)) = x(t)^T P x(t)$$

Stability of Linear Systems

SDP viewpoint:

- Lyapunov functions (generalize *energy* in systems). Functions on $x(t)$ decrease monotonically on trajectories of the system.
- For our discrete-time system, we have:

$$V(x(t)) = x(t)^T P x(t)$$

- To make these monotonically decreasing, we need:

$$\begin{aligned} V(x(t+1)) \leq V(x(t)) &\Leftrightarrow x(t+1)^T P x(t+1) - x(t)^T P x(t) \leq 0 \\ &\Leftrightarrow x(t)^T A^T P A x(t) - x(t)^T P x(t) \leq 0 \\ &\Leftrightarrow A^T P A - P \preceq 0 \end{aligned}$$

Stability of Linear Systems

SDP viewpoint:

- Lyapunov functions (generalize *energy* in systems). Functions on $x(t)$ decrease monotonically on trajectories of the system.
- For our discrete-time system, we have:

$$V(x(t)) = x(t)^T P x(t)$$

- To make these monotonically decreasing, we need:

$$\begin{aligned} V(x(t+1)) \leq V(x(t)) &\Leftrightarrow x(t+1)^T P x(t+1) - x(t)^T P x(t) \leq 0 \\ &\Leftrightarrow x(t)^T A^T P A x(t) - x(t)^T P x(t) \leq 0 \\ &\Leftrightarrow A^T P A - P \preceq 0 \end{aligned}$$

Theorem

Given matrix $A \in \mathbb{R}^{m \times m}$, the following conditions are equivalent:

- 1 All eigenvalues of A are inside unit circle, i.e. $|\lambda_i(A)| < 1$
- 2 There is $P \in \mathcal{S}^m$ such that

$$P \succ 0, \quad A^T P A - P \prec 0$$

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be $u(t) = Kx(t)$ for some fixed K (so that we use the system as its own feedback)

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be $u(t) = Kx(t)$ for some fixed K (so that we use the system as its own feedback)
- Same thing as replacing $A \leftarrow A + BK$

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be $u(t) = Kx(t)$ for some fixed K (so that we use the system as its own feedback)
- Same thing as replacing $A \leftarrow A + BK$
- Now this is harder to solve via simple eigenvalue description. But still solved the same way via Lyapunov functions!

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be $u(t) = Kx(t)$ for some fixed K (so that we use the system as its own feedback)
- Same thing as replacing $A \leftarrow A + BK$
- Now this is harder to solve via simple eigenvalue description. But still solved the same way via Lyapunov functions!
- Want $P \succ 0$ such that

$$(A + BK)^T P (A + BK) - P \prec 0$$

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

- If we properly choose control input $u(t)$ we can make our system $x(t)$ behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be $u(t) = Kx(t)$ for some fixed K (so that we use the system as its own feedback)
- Same thing as replacing $A \leftarrow A + BK$
- Now this is harder to solve via simple eigenvalue description. But still solved the same way via Lyapunov functions!
- Want $P \succ 0$ such that

$$(A + BK)^T P (A + BK) - P \prec 0$$

- Wait, this ain't no SDP! But we can make it into SDP with some matrix manipulations.

Conclusion

- Mathematical programming - very general, and pervasive in Algorithmic life
- General mathematical programming very hard

Conclusion

- Mathematical programming - very general, and pervasive in Algorithmic life
- General mathematical programming very hard
- Special cases have very striking applications!
 - Linear Programming (previous lectures)
 - Today: *Semidefinite Programming*

Conclusion

- Mathematical programming - very general, and pervasive in Algorithmic life
- General mathematical programming very hard
- Special cases have very striking applications!
 - Linear Programming (previous lectures)
 - Today: *Semidefinite Programming*
- Semidefinite Programming and Duality - fundamental concepts, lots of applications!
 - Applications in Combinatorial Optimization (Max-Cut in *next lecture!*)
 - Applications in Control Theory
 - many more!

Conclusion

- Mathematical programming - very general, and pervasive in Algorithmic life
- General mathematical programming very hard
- Special cases have very striking applications!
 - Linear Programming (previous lectures)
 - Today: *Semidefinite Programming*
- Semidefinite Programming and Duality - fundamental concepts, lots of applications!
 - Applications in Combinatorial Optimization (Max-Cut in *next lecture!*)
 - Applications in Control Theory
 - many more!
- Check out connections to Sum of Squares and a **bold**² attempt to have one algorithm to solve all problems! (i.e., one algorithm to rule them all)

<https://windowsontheory.org/2016/08/27/>

[proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/](#)

²pun intended

Acknowledgement

- Lecture based largely on:
 - [Blekherman, Parrilo, Thomas 2012, Chapter 2]

References I



Blekherman, Grigoriy and Parrilo, Pablo and Thomas, Rekha (2012)

Convex Algebraic Geometry