

Lecture 11: Linear Programming and Duality Theorems

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Overview

- Part I
 - Why Linear Programming?
 - Structural Results on Linear Programming
 - Duality Theory
- Conclusion
- Acknowledgements
- Proof of Fundamental Theorem of Linear Inequalities

Mathematical Programming

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- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$f(\mathbf{x}) = c_1 \cdot x_1 + \dots + c_n \cdot x_n + b = \mathbf{c}^T \mathbf{x} + b$$

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We can *always* represent LPs in *standard form*:

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0\end{array}$$

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$$\begin{array}{ll}\text{maximize} & p_1 \cdot x_1 + \cdots + p_n \cdot x_n \\ \text{subject to} & c_1 \cdot x_1 + \cdots + c_n \cdot x_n \leq B \\ & x \geq 0\end{array}$$

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- Other problems, such as *data fitting*, *linear classification* can be modelled as linear programs.

Important Questions

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- ④ How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs?

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Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \dots, a_m, b \in \mathbb{R}^n$, and $t := \text{rank}\{a_1, \dots, a_m, b\}$. Then either

- ① b is a **non-negative linear combination** of linearly independent vectors from a_1, \dots, a_m , or
- ② there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.
 - $c^T b < 0$
 - $c^T a_i \geq 0$
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Remark

Any hyperplane H with the above property is known as a **separating hyperplane**.

Farkas' Lemma

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

- 1 There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$
- 2 $y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$

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Equivalent formulation

Lemma (Farkas Lemma - variant 1)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

- 1 There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$
- 2 There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$

Farkas' Lemma

Equivalent formulation

Lemma (Farkas Lemma - variant 2)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

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- Let $M = [I \ A \ -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$

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- Let $M = [I \ A \ -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$
 - Now apply the original version of the lemma

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\begin{aligned}y^T A \leq c^T &\Rightarrow y^T Ax \leq c^T x \\ &\Rightarrow y^T b \leq c^T x\end{aligned}$$

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- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

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Primal LP

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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

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- When is the above inequality tight?

Strong Duality

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Theorem (Strong Duality)

If primal LP and dual LP are feasible, then

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- ① Since we have proved weak duality, suffices to show that the following LP has a solution:

$$\begin{array}{ll}\text{maximize} & 0 \\ \text{subject to} & y^T A \leq c^T \\ & c^T x - y^T b \leq 0 \\ & Ax = b \\ & x \geq 0\end{array}$$

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- ② Apply variant 2 of Farkas' lemma on the system above.

Proof of Strong Duality

① LP from previous page encoded by:

$$B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & A^T \\ -I & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ 0 \\ c \\ 0 \end{pmatrix}$$

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- ② Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \ v^T \ \lambda \ w^T \ e^T) \geq 0$ such that $zB = 0$ then we have $u^T b - v^T b + w^T c \geq 0$

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- ③ If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so $\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$

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- ② Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \ v^T \ \lambda \ w^T \ e^T) \geq 0$ such that $zB = 0$ then we have $u^T b - v^T b + w^T c \geq 0$
- ③ If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so
- $$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$
- ④ If $\lambda = 0$, let x, y be feasible solutions (which we assumed to exist). Then $x \geq 0, Ax = b$ and $y^T A \leq c^T$. Thus

$$c^T w \geq y^T Aw = 0 \geq (v^T - u^T)Ax = (v^T - u^T)b$$

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \leq b$$

have at least one solution, and suppose that inequality

$$c^T x \leq \delta$$

holds whenever x satisfies $Ax \leq b$. Then, for some $\delta' \leq \delta$ the linear inequality

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Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Complementary Slackness

- If the optima in both primal and dual is finite, and x, y are feasible solutions, the following are equivalent:
 - ① x, y are optimal solutions to the primal and dual
 - ② $c^T x = y^T b$
 - ③ if $x_i > 0$ then the corresponding inequality $y^T A_i \leq c_i$ is an equality: that is, we must have $y^T A_i = c_i$.

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- 2 and 3 are equivalent as we can write

$$c^T x - y^T b = c^T x - y^T A x = (c^T - y^T A) x = \sum_{i=1}^n (c_i - y^T A_i) x_i$$

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- General mathematical programming very hard (how hard do you think it is?)
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Today: *Linear Programming*

- Linear Programming and Duality - fundamental concepts, lots of applications!
 - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
 - Applications in Game Theory (minimax theorem)
 - Applications in Learning Theory (boosting)
 - many more

Acknowledgement

- Lecture based largely on:
 - [Schrijver 1986, Chapter 7]

Proof of Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \dots, a_m, b \in \mathbb{R}^n$, and $t := \text{rank}\{a_1, \dots, a_m, b\}$. Then either

- ① b is a *non-negative linear combination* of linearly independent vectors from a_1, \dots, a_m , or
- ② there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.
 - $c^T b < 0$
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- We will perform an iterative procedure:

Proof of Fundamental Theorem of Linear Inequalities

Iterative procedure, starting with \mathcal{L}_0 :

- 1 Write $b = \lambda_{i_1} a_{i_1} + \dots + \lambda_{i_n} a_{i_n}$. If $\lambda_i \geq 0$ we are done

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- To conclude the proof, need to show that this procedure always terminates. If process doesn't terminate, there are two times $r < t$ such that $\mathcal{L}_r = \mathcal{L}_t$
 - Let ℓ be the highest index for which a_ℓ has been removed from \mathcal{L}_k for some $r \leq k < t$.

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 - Let ℓ be the highest index for which a_ℓ has been removed from \mathcal{L}_k for some $r \leq k < t$.
 - $\mathcal{L}_r = \mathcal{L}_t \Rightarrow a_\ell$ has also been added from some $\mathcal{L}_{k'}$ for some $r \leq k' < t$.

Proof of Fundamental Theorem of Linear Inequalities

- Say a_r was removed at iteration k and added back at iteration k' so $r \leq k < k' < t$
- Let c be the vector defining the hyperplane at the k' iteration (when we added a_r back to the set), and let $\mathcal{L}_k = \{a_{i_1}, \dots, a_{i_n}\}$
- Now, above implies the following contradiction:

$$0 > c^T b = c^T (\lambda_{i_1} a_{i_1} + \dots + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} c^T a_{i_1} + \dots + \lambda_{i_n} c^T a_{i_n} \geq 0$$

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- Second inequality holds term by term:
 -

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