Lecture 9: Random Walks, Markov Chains, Mixing Time

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Overview

- Introduction
  - Why Random Walks & Markov Chains?
  - Basics on Theory of Finite Markov Chains
- Main Topics
  - Stationary Distributions and Mixing Time
  - Fundamental Theorem of Markov Chains
- Acknowledgements

What is a Random Walk?

Given a graph $G(V, E)$

1. random walk starts from a vertex v_0
2. at each time step it moves *uniformly* to a *random neighbor* of the current vertex in the graph

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- *Cover time:* how long does it take to reach every vertex of the graph at least once?

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- **Practice question:** Compare question 2 to coupon collector problem!

What is a Markov Chain?

Random walk is a special kind of *stochastic process*:

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Process is "*forgetful*/*memoryless*"

Markov chain is characterized by this property.

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- Markov Chain *irreducible* if underlying directed graph is *strongly connected* (i.e. there is directed path from i to j for any pair $i, j \in V$)

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- Transition given by

$$p_{t+1} = P \cdot p_t$$

Properties of Markov Chains

- *Period* of a state i is:

$$\gcd\{t \in \mathbb{N} \mid P_{i,i}^t > 0\}$$

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Lemma

For any finite, irreducible and aperiodic Markov Chain, there exists $T < \infty$ such that

$$P^t_{i,j} > 0 \quad \text{for any } i, j \in V \text{ and } t \geq T.$$

See proof in reference of [Häggström, Chapter 4].

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- Given two distributions $p, q \in \mathbb{R}^n$, their *total variational distance* is

$$\Delta_{TV}(p, q) = \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2} \cdot \|p - q\|_1$$

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- p_t *converges* to q iff $\lim_{t\to\infty}\Delta_{TV}(p_t, q) = 0$

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- For complete graph, eigenvalues $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_n = -1/(n-1)$, corresponding eigenvectors $v_1, \ldots, v_n$ (orthonormal)

Hitting Time

- Given states i, j in a Markov chain, the *hitting time* from state i to state j is defined as

$$T_{i,j} := \min\{t \geq 1 \mid X_t = j, X_0 = i\}$$

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- *Hitting time lemma*: For any *finite*, *irreducible*, *aperiodic* Markov chain, and for any two states i, j (not necessarily distinct) we have that:

$$\Pr[T_{i,j} < \infty] = 1 \quad \text{and} \quad \mathbb{E}[T_{i,j}] < \infty$$

Proof of Hitting Time Lemma

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- Note that

$$\Pr[T_{i,j} > M] \leq \Pr[X_M \neq j] \leq 1 - \alpha$$

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- Moreover, we can prove:

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- Thus, we have

$$\mathbb{E}[T_{i,j}] = \sum_{n \geq 1} \Pr[T_{i,j} \geq n] = \sum_{n \geq 0} \Pr[T_{i,j} > n] \leq M/\alpha < \infty$$

Fundamental Theorem of Markov Chains

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Theorem (Fundamental Theorem of Markov Chains)

Any finite, irreducible and aperiodic Markov Chain has the following properties:

1. *There exists a unique stationary distribution π, where $\pi_i > 0$ for all $i \in [n]$*
2. *The sequence of distributions $\{p_t\}_{t \geq 0}$ will converge to π, no matter what the initial distribution is*
3. $$\pi_i = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{\tau_{i,i}}$$

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- Note that in this case, easy to guess stationary distribution:

$$\pi_i = \frac{d_i}{2m}, \quad m = |E|$$

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 6]
  - [Häggström]
- See Lap Chi's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf
- Also see Lap Chi's notes https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf for a proof of fundamental theorem of Markov chains for undirected graphs.

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