

Lecture 07: Algebraic Techniques

Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

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Overview

- Introduction

- Why Algebraic Techniques in computer science?
- Fingerprinting: String equality verification

- Main Problems

- Polynomial Identity Testing
- Randomized Matching Algorithms
- Isolation Lemma

- Remarks

- Acknowledgements

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Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

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Communication complexity setting, randomized algorithms, need to work with high probability.

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- what happens when they are different?

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- Number of bits sent is $\tilde{O}(\log t + \log n)$. Choosing $t = n$ solves it.

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- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

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- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

- Can amplify probability by running multiple times or by choosing larger set S .

Polynomial Identity Lemma

Lemma (Polynomial Identity Lemma)

Let $\mathbb{F}$ be a field and $P(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ be a **nonzero** polynomial of degree $\leq d$. Then for any set $S \subseteq \mathbb{F}$, we have:

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Proof by induction in number of variables.

- Introduction

- Why Algebraic Techniques in computer science?
- Fingerprinting: String equality verification

- Main Problems

- Polynomial Identity Testing
- Randomized Matching Algorithms
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Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?

²First proved by Edmonds.

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- **Algorithm:** evaluate $\det(X)$ at a random value for the variables $y_{i,j}$.

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We have seen randomized algorithms for bipartite and non-bipartite matching.

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How? See the (future notes) in my CS 860 course.

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Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \dots, n\}$, if we assign a random weight function $w : [n] \rightarrow [2n]$ then the probability that there is a unique minimum weight set is at least $1/2$.

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 $w(1) = 3$, $w(2) = 5$, $w(3) = 8$, $w(4) = 4$

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Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is **only one** set of minimum weight.

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10 Probability that this happens is $\leq 1/2$. (step 8)

Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
 - Prof. Lau's notes
 - [Motwani & Raghavan 2007, Chapter 7]
 - [Korte & Vygen 2012, Chapter 10].
- See Prof. Lau's notes at
<https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf>

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 Fenner, Stephen and Gurjar, Rohit and Thierauf, Thomas (2019)

Bipartite perfect matching is in quasi-NC.

SIAM Journal on Computing

 Svensson, Ola and Jakub Tarnawski (2017)

The matching problem in general graphs is in quasi-NC.

IEEE 58th Annual Symposium on Foundations of Computer Science

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- Each vertex in H_σ has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.

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- If σ only has even cycles, then H_σ gives us a perfect matching (by taking every other edge of the graph H_σ , ignoring orientation)

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- Otherwise, for each $\sigma \in S_{2n}$ (that has odd cycle), there is another permutation $r(\sigma) \in S_{2n}$ that is obtained by reversing odd cycle of H_σ containing vertex with *minimum index*.

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$$(-1)^\sigma \prod_{i=1}^{2n} [T_G]_{i, \sigma(i)} = (-1)^n \prod_{i=1}^n -x_{(2i-1)\sigma(2i-1)}^2 = \prod_{i=1}^n x_{(2i-1)\sigma(2i-1)}^2.$$