

Lecture 5: Hashing

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Overview

- Introduction
 - Hash Functions
 - Why is hashing?
 - How to hash?
- Succinctness of Hash Functions
 - Coping with randomness
 - Universal Hashing
 - Hashing using 2-universal families
 - Perfect Hashing
- Acknowledgements

Computational Model

Before we talk about hash functions, we need to state our model of computation:

Definition (Word RAM model)

In the word RAM^a model:

- all elements are integers that fit in a machine word of w bits
- Basic operations (comparison, arithmetic, bitwise) on such words take $\Theta(1)$ time
- We can also access *any* position in the array in $\Theta(1)$ time

^aRAM stands for Random Access Model

What is hashing?

We want to store ℓ elements (keys) from the set $U = \{0, 1, \dots, m - 1\}$, where $2^w > m \gg \ell$, in a data structure that supports *insertions*, *deletions*, *search* “as efficiently as possible.”

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- Memory: $\Theta(m)$ *(this is very bad!)*

Want to also achieve optimal memory $O(\ell)$. For this we will use a technique called *hashing*.

- A *hash function* is a function $h : U \rightarrow [0, n-1]$, where $|U| = m \gg n$.
- A *hash table* is a data structure that consists of:
 - a table T with n cells $[0, n-1]$,
 - a hash function $h : U \rightarrow [0, n-1]$

From now on, we will define memory as *# of cells*.

Why is hashing useful?

- Designing efficient data structures (dictionaries) for searching
- Data streaming algorithms
- Derandomization
- Cryptography
- Complexity Theory
- many more

Challenges in Hashing

Setup:

- Universe $U = \{0, \dots, m - 1\}$ of size $m \gg n$ where n is the size of the range of our hash function $h : U \rightarrow [0, n - 1]$
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Ideally, want hash function to map *different keys* into *different locations*.

Definition (Collision)

We say that a *collision* happens for hash function h with inputs $x, y \in U$ if $x \neq y$ and $h(x) = h(y)$.

By pigeonhole principle, impossible to achieve without knowing keys in advance.

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Will settle for: $\#$ collisions *small with high probability*.

Our solution: family of hash functions

Construct *family* of hash functions $\mathcal{H}$ such that the *number of collisions* is **small** with **high probability**, when we pick hash function uniformly at random from the family $\mathcal{H}$.

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Simplest version to keep in mind:

$$\Pr_{h \in \mathcal{H}} [h(x) = h(y)] \leq \frac{1}{\text{poly}(n)} \quad \forall x \neq y \in U$$

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Assumptions:

- keys are independent from hash function we choose.
- we **do not** know keys in advance (even if we did, nontrivial problem!)

Question

Still could have collisions. How do we handle them?

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- Expected number of keys in a location: 1
- maximum number of collisions (max load) in one particular location: $O(\log n / \log \log n)$ keys

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Could also pick *two* random hash functions and use *power of two choices*.

Collision bound becomes $O(\log \log n)$.

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How do we cope with the computational problem that arose with randomness?

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Part of *derandomization/pseudorandomness*: huge subfield in TCS!

k -wise independence

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Definition (Full Independence)

A set of random variables $X_1, \dots, X_n$ are said to be (fully) independent if they satisfy

$$\Pr \left[\bigcap_{i=1}^n X_i = a_i \right] = \prod_{i=1}^n \Pr[X_i = a_i]$$

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Definition (k -wise Independence)

A set of random variables $X_1, \dots, X_n$ are said to be k -wise independent if for any set $J \subset [n]$ such that $|J| \leq k$ they satisfy

$$\Pr \left[\bigcap_{i \in J} X_i = a_i \right] = \prod_{i \in J} \Pr[X_i = a_i]$$

Pairwise independence

When $k = 2$, k -wise independence is called *pairwise independence*.

Example (XOR pairwise independence)

Given t uniformly random bits $Y_1, \dots, Y_t$, we can generate $2^t - 1$ pairwise independent random variables as follows:

$$X_S := \bigoplus_{i \in S} Y_i \quad S \subseteq [t] \setminus \emptyset$$

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Pairwise independence II

Example (Pairwise independence in $\mathbb{F}_p$)

Let p be a prime number. Given 2 uniformly random variables $Y_1, Y_2 \sim [0, \dots, p-1]$, generate p pairwise independent random variables as follows:

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Can think of these random variables as picking a random line over a finite field. If we only know one point of the line, the second point is still uniformly random. However two points determine the line.

Universal Hash Functions

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Let U be a universe with $|U| \geq n$. A family of hash functions $\mathcal{H} = \{h : U \rightarrow [0, n-1]\}$ is *k-universal* if, *for any distinct* elements $u_1, \dots, u_k \in U$, we have

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Definition (Strongly Universal Hash Functions)

$\mathcal{H} = \{h : U \rightarrow [0, n-1]\}$ is ***strongly k-universal*** if, *for any distinct* elements $u_1, \dots, u_k \in U$ and *for any* values $y_1, \dots, y_k \in [0, n-1]$, we have

$$\Pr_{h \in_R \mathcal{H}} [h(u_1) = y_1, \dots, h(u_k) = y_k] = 1/n^k$$

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Can use random variables to construct universal hash functions!

Strongly 2-universal families of hash functions

Let p be a prime number, $U = [0, p - 1]$.

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := a \cdot x + b \bmod p \mid a, b \in [0, p - 1]\}$$

is strongly 2-universal.

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Proposition

Let $U = [0, p^k - 1] \equiv [0, p - 1]^k \setminus \{(0, \dots, 0)\}$ and $\vec{a} = (a_0, \dots, a_{k-1})$

$$\mathcal{H} = \{h_{\vec{a},b}(\vec{x}) := \vec{a} \cdot \vec{x} + b \pmod{p} \mid \vec{a} \in U, b \in [0, p - 1]\}$$

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2-universal families of hash functions

What if my hash table size is not a prime?

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := (a \cdot x + b \bmod p) \bmod n \mid a, b \in [0, p-1], a \neq 0\}$$

is 2-universal (but not strongly 2-universal).

Practice problem: prove the proposition above.

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- Two points determine a line. Similarly, k points determine a univariate polynomial of degree $k - 1$
- Random degree $k - 1$ polynomials are k -wise independent!
- Practice problem: prove this!

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- In XOR example, our function takes $O(t)$ storage space, and $O(t)$ time to compute.^a
- In $\mathbb{F}_p$ examples, our function takes $O(1)$ storage space and $O(1)$ time to compute!^b

^aReminder that we assume that $t < w$.

^bWe assume that $p < 2^w$.

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Lemma (Maximum number of collisions)

Let our set of keys S be of size ℓ , and our hash functions be from U to $[0, n - 1]$. The expected number of collisions, using a 2-universal hash family is

$$\leq \ell^2 / 2n$$

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Thus, by Markov's inequality, we have

Lemma (Maximum load of entry of hash table)

With probability $\geq 1/2$ the maximum load of a cell of a hash table, when using a 2-universal hash family is

$$\leq \sqrt{\frac{2\ell^2}{n}}.$$

When $\ell \approx n$ (as is usually assumed in hashing), we expect $\sqrt{2n}$.

Perfect Hashing

Setting: (*static keys*) Suppose now we are given the set S of keys in advance, and $|S| = n$ (so, $\ell = n$ here).

How to build a hash table with $O(1)$ search time and $O(n)$ memory? Can we still do it with a 2-universal family of hash functions?

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Corollary

If $h \in \mathcal{H}$ is a random hash function from a 2-universal family of hash functions, then for any set $S \subseteq U$ of size $\ell \leq \sqrt{n}$, the probability of h being perfect for S is at least $1/2$.

Proof: There is no collision with probability $\geq 1/2$.

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New idea: build a *two-level* hash table!

Perfect Hashing

Setting: (*static keys*) Suppose now we are given the set S of keys in advance, and $|S| = n$ (so, $\ell = n$ here).

How to build a hash table with $O(1)$ search time and $O(n)$ memory? Can we still do it with a 2-universal family of hash functions?

Corollary

If $h \in \mathcal{H}$ is a random hash function from a 2-universal family of hash functions, then for any set $S \subseteq U$ of size $\ell \leq \sqrt{n}$, the probability of h being perfect for S is at least $1/2$.

Proof: There is no collision with probability $\geq 1/2$.

New idea: build a *two-level* hash table!

Theorem

The two-level approach gives perfect hashing scheme.

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's notes at
<https://cs.uwaterloo.ca/~lapchi/cs466/notes/L05.pdf>

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